A Memoir on the Abelian and Theta Functions.

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The present memoir is based upon Clebsch and Gordan's "*Theorie der Abelschen Functionen,*" Leipzig, 1866 (here cited as C. and G.); the employment of differential rather than of integral equations is a novelty; but the chief addition to the theory consists in the determination which I have made for the cubic curve, and also (but not as yet in a perfect form) for the quartic curve, of the differential expression $d\Pi_{\xi\eta}$ (or as I write it $d\Pi_{12}$) in the integral of the third kind $\int_\alpha^\beta d\Pi_{\xi\eta}$ in the final normal form (endliche Normalform) for which we have (p. 117) $\int_\xi^\eta d\Pi_{\alpha\beta} = \int_\alpha^\beta d\Pi_{\xi\eta}$, the limits and parametric points interchangeable. The want of this determination presented itself to me as a *lacuna* in the theory during the course of lectures on the subject which I had the pleasure of giving at the Johns Hopkins University, Baltimore, U. S. A., in the months January to June, 1882, and I succeeded in effecting it for the cubic curve, but it was not until shortly after my return to England that I was able partially to effect the like determination in the far more difficult case of the quartic curve. The memoir contains, with additional developments, a reproduction of the course of lectures just referred to. I have endeavored to simplify as much as possible the notations and demonstrations of Clebsch and Gordan's admirable treatise; to bring some of the geometrical results into greater prominence; and to illustrate the theory by examples in regard to the cubic, the nodal quartic, and the general quartic curves respectively. The present three chapters are: I, Abel's Theorem; II, Proof of Abel's Theorem; III, The Major Function. The paragraphs of the whole memoir will be numbered continuously.

CHAPTER I. ABEL'S THEOREM.

The Differential Pure and Affected Theorems. Art. Nos. 1 to 5.

1. We have a fixed curve and a variable curve, and the differential pure theorem consists in a set of linear relations between the displacements of the intersections of the two curves; in the affected theorem a linear function of the displacements is equated to another differential expression. I state the two theorems, giving afterwards the necessary explanations.

The pure theorem is

$$\Sigma\,(x,\,y,\,z)^{n-3}d\omega = 0\,.$$

The affected theorem is

$$\Sigma\,\frac{(x,\,y,\,z)_{12}^{n-2}d\omega}{012} = -\,\frac{\delta\varphi_1}{\varphi_1} + \frac{\delta\varphi_2}{\varphi_2} \qquad \text{(See footnote.*)}$$

2. We have a fixed curve $f=0$, or say the curve f, or simply the fixed curve, of the order n, with δ dps, and therefore of the deficiency $\frac{1}{2}(n-1)(n-2) - \delta,\ = p$. The expression the dps means always the δ dps of f.

And we have a variable curve $\phi=0$, or say the curve ϕ, or simply the variable curve, of the order m, passing through the dps and besides meeting the fixed curve in $mn-2\delta$ variable points.

Moreover, $d\omega$ is the displacement of the current point 0, coordinates (x, y, z), on the fixed curve, viz. the equation $f=0$ gives

$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0\,,$$

$$\frac{df}{dx}x + \frac{df}{dy}y + \frac{df}{dz}z = 0\,,$$

and we thence have

$$\frac{df}{dx} : \frac{df}{dy} : \frac{df}{dz} = ydz - zdy : zdx - xdz : xdy - ydx\,,$$

so that we have three equal values each of which is put $= d\omega$, viz. we write

$$\frac{ydz - zdy}{\frac{df}{dx}} = \frac{zdx - xdz}{\frac{df}{dy}} = \frac{xdy - ydx}{\frac{df}{dz}}\,,\ = d\omega\,,$$

and $d\omega$ as thus defined is the displacement.

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* For comparison with C. and G. observe that in the equation, p. 47, $V = \log\frac{\psi(\eta)\,\phi(\xi)}{\phi(\eta)\,\psi(\xi)},\ = \log\frac{\psi_2\phi_1}{\phi_2\psi_1}$ suppose, their ψ belongs to the upper limit and corresponds to my ϕ: the equation gives therefore $dV = -\delta\frac{\psi_1}{\psi_1} + \delta\frac{\psi_2}{\psi_2}$, agreeing with the formula in the text.

$(x, y, z)^{n-3} = 0$ is the minor curve, viz. the general curve of the order $n-3$, which passes through the dps;* and the function $(x, y, z)^{n-3}$ is the minor function.

1 and 2 are fixed points on f, called the parametric points, coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; and 012 denotes the determinant

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

so that $012 = 0$ is the equation of the line joining the points 1 and 2: this line meets the fixed curve in $n-2$ other points, called the residues of 1, 2.

$(x, y, z)_{12}^{n-2} = 0$ is the major curve *quoad* the points 1 and 2; viz. this is the general curve of the order $n-2$, passing through the dps and also through the residues of 1, 2.

But further, the function $(x, y, z)_{12}^{n-2}$ is the proper major function; viz. the implicit factor of the function is so determined that taking $0 = 1$, the current point at 1 (that is writing (x_1, y_1, z_1) for (x, y, z)) the function $(x, y, z)_{12}^{n-2}$ reduces itself to the polar function $\left(x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1}\right) f_1$, afterwards written $n.1^{n-1}2$, of f: this implies that taking $0 = 2$, the current point at 2, the function reduces itself to the polar function $n.12^{n-1}$.

ϕ_1 is what ϕ becomes on writing therein (x_1, y_1, z_1) for (x, y, z): and similarly ϕ_2 is what ϕ becomes on writing therein (x_2, y_2, z_2) for (x, y, z).

δ denotes differentiation in regard only to the coefficients of ϕ; viz. writing $\phi = (\mathrm{a}, \ldots \between x, y, z)^m$ we have $\delta\phi = (d\mathrm{a}, \ldots \between x, y, z)^m$, and similarly $\delta\phi_1$ and $\delta\phi_2 = (d\mathrm{a}, \ldots \between x_1, y_1, z_1)^m$ and $(d\mathrm{a}, \ldots \between x_2, y_2, z_2)^m$ respectively.

The sum Σ extends to all the variable intersections of the two curves.

3. As to the meaning of the theorems, consider first the pure theorem. The variable intersections are not all of them arbitrary points on the fixed curve: a certain number of them taken at pleasure on the fixed curve will determine the remaining variable intersections; and there are thus a certain number of integral relations between the coordinates of the variable intersections; to each such integral relation there corresponds a linear relation between the displacements $d\omega$ of these points, or say a displacement-relation. It is precisely these displacement-relations which are given by the theorem, viz. the equation

$$\Sigma (x, y, z)^{n-3} d\omega = 0$$

* This definition implies that the number of dps is at most $= \frac{1}{2}(n-1)(n-2) - 1$, that is that the fixed curve is not unicursal. But see *post* No. 21.

breaks up into as many linear relations as there are constants in the function $(x, y, z)^{n-3}$ which equated to zero gives a curve of the order $n-3$ passing through the dps; for instance $n=3$, $\delta=0$, the equation gives the single relation $\Sigma d\omega=0$; but $n=4$, $\delta=0$, the equation gives the three relations $\Sigma x d\omega=0$, $\Sigma y d\omega=0$, $\Sigma z d\omega=0$.

4. It is of course important to show, and it will be shown, that the number of independent displacement-relations given by the theorem is equal to the number of independent integral relations between the variable intersections.

5. Observe that the pure theorem gives *all* the displacement-relations between the variable intersections; we are hereby led to see the nature of the affected theorem. Taking at pleasure on the fixed curve the sufficient number of variable intersections, the coefficients of ϕ are thereby determined in terms of the coordinates of the assumed variable intersections,* and hence the value of $-\dfrac{\delta\varphi_1}{\varphi_1}+\dfrac{\delta\varphi_2}{\varphi_2}$ is given as a linear function of the corresponding displacements $d\omega$; and, substituting this value, the affected theorem gives a linear relation between the displacements $d\omega$ of the several variable intersections. But any such linear relation must clearly be a mere linear combination of the displacement-relations $\Sigma(x, y, z)^{n-3}\, d\omega=0$ given by the pure theorem.

Examples of the Pure Theorem—The Fixed Curve a Cubic. Art. Nos. 6 to 12.

6. The pure theorem is not applicable to the case $n=2$, the fixed curve a conic: it in fact gives no displacement-relation; and this is as it should be, for the variable intersections are all of them arbitrary.

The next case is $n=3$, $\delta=0$, the fixed curve a cubic. For greater simplicity the equation is taken in Cartesian coordinates. In general for such an equation, writing in the homogeneous formulæ $z=1$, we have

$$d\omega=\frac{dx}{\dfrac{df}{dy}}=-\frac{dy}{\dfrac{df}{dx}},$$

(the two values being of course equal in virtue of $\dfrac{df}{dx}dx+\dfrac{df}{dy}dy=0$); taking the former value and considering $\dfrac{df}{dy}$ as expressed in terms of x, let this be called

* The coefficients are determined, except it may be as to some constants which remain arbitrary, but which disappear from the difference $-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}$; this will be explained further on in the text.

P (of course P is an irrational function of x): then we have $d\omega = \frac{dx}{P}$; and similarly $d\omega_1 = \frac{dx_1}{P_1}$, etc.

The fixed curve being then a cubic, let the variable curve be a line; this meets the cubic in three points, say $1, 2, 3$; and any two of these determine the line, and therefore the third point; there should therefore be one integral relation, and consequently one displacement-relation; and this is what is given by the theorem, viz. we have $\Sigma d\omega = 0$, that is $d\omega_1 + d\omega_2 + d\omega_3 = 0$, or what is the same thing

$$\frac{dx_1}{P_1} + \frac{dx_2}{P_2} + \frac{dx_3}{P_3} = 0.$$

The corresponding integral equation is the equation which expresses that the points $1, 2, 3$ are in a line, viz. considering y_1, y_2, y_3 as given functions of x_1, x_2, x_3 respectively, this is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

or, in the notation already made use of for such a determinant, $123 = 0$.

7. This equation $d\omega_1 + d\omega_2 + d\omega_3 = 0$, where $d\omega$ denotes $\frac{dx}{P}$, has a peculiar interpretation when we consider the coefficients of the cubic as arbitrary constants, and therefore the cubic as a curve depending upon nine arbitrary constants.* In taking 1 a point on the curve we in effect determine y_1 as a function of x_1 and the nine constants; and similarly in taking 2 a point on the curve we determine y_2 as a function of x_2 and the nine constants; the points 1 and 2 determine the third intersection 3, and we have thus x_3 determined as a function of x_1, x_2 and the nine constants.

Considering x_3 as thus expressed, we have $dx_3 = \frac{dx_3}{dx_1}dx_1 + \frac{dx_3}{dx_2}dx_2$, an equation which must agree with $d\omega_1 + d\omega_2 + d\omega_3 = 0$, that is with $dx_3 = -\frac{P_3}{P_1}dx_1 - \frac{P_3}{P_2}dx_2$. It follows that we have $\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2} = \frac{P_2}{P_1}$, and taking the logarithms and differentiating with $\frac{d}{dx_1}\cdot\frac{d}{dx_2}$ we find $\frac{d}{dx_1}\frac{d}{dx_2}\log\left(\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2}\right) = 0$, a partial differential equation of the third order, independent of any particular

* This theory was communicated by me to Section A of the British Association at the York meeting. See B. A. Report, 1881, pp. 534–535, "A Partial Differential Equation connected with the Simplest Case of Abel's Theorem."

cubic curve, and satisfied by x_3 considered as a function of x_1, x_2 and the nine constants. Observe that starting from the expression for x_3, and proceeding to the differential coefficients of the third order, we have ten equations from which the nine constants can be eliminated, that is we ought to have a partial differential equation of the third order: and conversely that the equation for x_3, as containing nine arbitrary constants, is a complete solution of the partial differential equation: the complete solution of the partial differential equation in question is thus the equation which expresses that 3 is the remaining intersection of the line through 1 and 2 with a cubic.

8. The partial differential equation has a geometrical interpretation, or is at least very closely connected with a geometrical property. Consider three consecutive positions of the line, meeting the cubic in the points 1, 2, 3; 1′, 2′, 3′ and 1″, 2″, 3″ respectively: the three lines constitute a cubic curve: the nine points are thus the intersections of two cubic curves, or say they are an "ennead" of points: and any eight of the points thus determine uniquely the ninth point.

9. As a particular example let the cubic be $x^3+y^3-1=0$; then $y=\sqrt[3]{1-x^3}$, and $d\omega=\dfrac{dx}{y^2}, =\dfrac{dx}{(1-x^3)^{\frac{2}{3}}}$ *; and with these values we have as before the differential relation $d\omega_1+d\omega_2+d\omega_3=0$, and the integral relation $123=0$. I give a direct verification. To find x_3, y_3 the coordinates of the third intersection, we may in the equation of the cubic write $x_3, y_3, 1=\lambda x_1+\mu x_2, \lambda y_1+\mu y_2, \lambda+\mu$ respectively, and then writing for shortness $1^2 2=x_1^2x_2+y_1^2y_2-1$, $12^2=x_1x_2^2+y_1y_2^2-1$, we obtain for the determination of λ, μ the equation $\lambda . 1^2 2+\mu . 12^2=0$.

This being so, from the equation $123=0$, we obtain by differentiation

$$\Sigma\{dx_1(y_2-y_3)-dy_1(x_2-x_3)\}=0,$$

the sum consisting of three terms, the second and third of them being obtained from the one written down by the cyclical interchange of the numbers 1, 2, 3. But we have $x_1^2dx_1+y_1^2dy_1=0$, and the equation thus is

$$\Sigma\frac{dx_1}{y_1^2}\{y_1^2(y_2-y_3)+x_1^2(x_2-x_3)\}=0:$$

* Writing $f=x^3+y^3-1$ we should have $\frac{df}{dy}=3y^2$, and therefore $d\omega=\frac{dx}{3y^2}$; but the $\frac{1}{3}$ enters as a common factor in all the $d\omega$'s, and it may clearly be disregarded: the value in the text, $d\omega=\frac{dx}{y^2}$ could of course be obtained by writing, as we may do, $f=\frac{1}{3}(x^3+y^3-1)$, and so in other cases.

this will reduce itself to $\Sigma\,\dfrac{dx_1}{y_1^2}=0$, if only the three coefficients in $\{\,\}$ are equal, that is we ought to have

$$y_1^2\,(y_2-y_3)+x_1^2\,(x_2-x_3)=y_2^2\,(y_3-y_1)+x_2^2\,(x_3-x_1)=y_3^2\,(y_1-y_2)+x_3^2\,(x_1-x_2).$$

Comparing for instance the first and second terms, the equation is

$$-y_3\,(y_1^2+y_2^2)-x_3\,(x_1^2+x_2^2)+(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

or as this may be written

$$-(\lambda y_1+\mu y_2)(y_1^2+y_2^2)-(\lambda x_1+\mu x_2)(x_1^2+x_2^2)+(\lambda+\mu)(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

where the whole coefficient of λ is $-x_1^3-y_1^3+x_1^2x_2+y_1^2y_2$, which in fact is $x_1^2x_2+y_1^2y_2-1,=1^22$; and similarly the whole coefficient of μ is 12^2; the equation is thus $\lambda\,.\,1^22+\mu\,.\,12^2=0$, which is right. The first and second coefficients are thus equal, and in like manner the first and third coefficients are equal; we have thus the required result, $\dfrac{dx_1}{y_1^2}+\dfrac{dx_2}{y_2^2}+\dfrac{dx_3}{y_3^2}=0$.

10. In all that follows, the cubic might be any cubic whatever, but to fix the ideas I take a particular form.

Let the cubic be $y^2-X=0$, X a cubic function $(x,\,1)^3$, or say even $X=x\,.\,1-x\,.\,1-k^2x$, then $y=\sqrt{X}$, $d\omega=\dfrac{dx}{y}$, $=\dfrac{dx}{\sqrt{X}}$; and with these values we have the differential relation $d\omega_1+d\omega_2+d\omega_3=0$, and the integral relation $123=0$. This last equation is an integral of the differential equation $d\omega_1+d\omega_2+d\omega_3=0$; as not containing any arbitrary constant it is a particular integral.

But regard one of the three points, say 3, as a fixed point, that is let the line pass through the fixed point 3 of the cubic, and besides meet it in the points 1 and 2. We write $d\omega_3=0$, and the differential equation thus is $d\omega_1+d\omega_2=0$, while the integral equation is as before $123=0$; this equation, as containing one arbitrary constant, is the general integral of $d\omega_1+d\omega_2=0$.

Let the variable curve be a conic; say the intersections with the cubic are $1, 2, 3, 4, 5, 6$. Any five of these points determine the conic, and therefore the sixth point; there is thus one integral relation, the equation $123456=0$, which expresses that the six points are in a conic, and there should therefore be one displacement-relation, viz. this is the equation $\Sigma d\omega=0$, that is $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5+d\omega_6=0$.

We have thus $123456=0$, as a particular integral of $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5+d\omega_6=0$. If, however, we take 6 a fixed point on the cubic, then we have the same equation as the general integral of $d\omega_1+d\omega_2+d\omega_3+d\omega_4+d\omega_5=0$.

But taking also 5 a fixed point of the cubic we have as an integral of $d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 = 0$, the foregoing equation $123456 = 0$, which contains apparently two arbitrary constants; and so if we also fix the point 4, or the points 4 and 3, we have for the differential equations $d\omega_1 + d\omega_2 + d\omega_3 = 0$, and $d\omega_1 + d\omega_2 = 0$, integrals with apparently three arbitrary constants and four arbitrary constants respectively.

11. The explanation is contained in the theory of *Residuation* on a cubic curve. Take the case $d\omega_1 + d\omega_2 + d\omega_3 = 0$, with the integral $123456 = 0$, containing apparently three arbitrary constants, viz. the relation between the variable points 1, 2, 3, is here given by a construction depending on the three fixed points 4, 5, 6 on the cubic; it is to be shown that two of these points can always be regarded as no-matter-what* points. To see that this is so, take on the cubic any two no-matter-what points $4'$, $5'$, then according to the theory referred to, we can find on the cubic a determinate point $6'$ such that the points $4'$, $5'$ and $6'$ establish between the variable points 1, 2, 3, the same relation which is established between them by means of the points 4, 5 and 6, viz. whether in order to determine the point 3 we draw a conic through 1, 2, 4, 5 and 6; or a conic through 1, 2, $4'$, $5'$ and $6'$, we obtain as the remaining intersection of the conic with the cubic one and the same point 3. The construction of $6'$ is, through 4, 5 and 6 draw a conic meeting the cubic in any three points 1, 2, 3; through these points and $4'$, $5'$ draw a conic, the remaining intersection of this with the cubic will be the required point $6'$, and the point $6'$ thus obtained will be a determinate point, independent of the particular conic through 4, 5 and 6 used for the construction. Thus 4 and 5 are replaceable by the no-matter-what points $4'$ and $5'$, or what is the same thing, two of the points 4, 5 and 6 may be regarded as no-matter-what points, and the number of arbitrary constants is thus reduced to one. And so in other cases, all but one of the fixed points may be regarded as no-matter-what points, and the integral as containing in each case only one arbitrary constant.

But conversely, it being known that the integral of the differential equation contains but one arbitrary constant, we can thence arrive at the theory of residuation.

* The epithet explains, I think, itself; the point may be any point at pleasure, but it is quite immaterial what point, and for this reason it is not counted as an arbitrary point. The most simple instance is that of two constants presenting themselves in a combination such as $c + c'$, either of them may be regarded as a no-matter-what constant.

12. We might go on to the case where the variable curve is a cubic; there are here nine intersections; any eight of these do *not* determine the variable cubic, but they *do* determine the ninth intersection; and there is between the nine intersections one integral relation, and corresponding to it one displacement-relation $\Sigma d\omega = 0$, that is $d\omega_1 + d\omega_2 + \ldots + d\omega_9 = 0$, given by the pure theorem. But as to this see further on, where it is shown in general that the number of independent integral relations is equal to the number of independent displacement-relations given by the theorem.

Example of the Affected Theorem—Fixed Curve a Circle. Art. Nos. 13 and 14.

13. The fixed curve is taken to be the circle $x^2 + y^2 - 1 = 0$, and the parametric points 1 and 2 to be the points (1, 0) and (0, 1) on this circle. The variable curve is taken to be a line, say the line $ax + by - 1 = 0$, meeting the circle in the points 3 and 4, coordinates (x_3, y_3) and (x_4, y_4) respectively.

Starting from the formula

$$\Sigma \frac{(x, y, 1)^0_{12}\, d\omega}{012} = -\frac{\delta\varphi_1}{\varphi_1} + \frac{\delta\varphi_2}{\varphi_2},$$

where the summation extends to the points 3 and 4, $(x, y, 1)^0_{12}$ is here a constant, $= 2.12$, that is $2\,(x_1x_2 + y_1y_2 - 1)$, which for the points 1, 2 in question is $= -2$. We have 012 denoting the determinant

$$\left|\begin{array}{ccc} x, & y, & 1 \\ 1, & 0, & 1 \\ 0, & 1, & 1 \end{array}\right|,$$

which is $= -x - y + 1$; and $d\omega = \dfrac{dx}{2y}\cdot$ Also $\dfrac{\delta\varphi_1}{\varphi_1}, = \dfrac{xda + ydb}{ax + by - 1}$, is $= \dfrac{da}{a-1}$, and similarly $\dfrac{\delta\varphi_2}{\varphi_2}$ is $= \dfrac{db}{b-1}\cdot$ The formula thus is

$$\Sigma \frac{dx}{y(x + y - 1)} = -\frac{da}{a-1} + \frac{db}{b-1}.$$

The coefficients a and b are determined by means of the points 3 and 4, that is they are functions of x_3, x_4; and considering them as thus expressed, then (inasmuch as there is no linear relation between the displacements $\dfrac{dx_3}{y_3}$ and $\dfrac{dx_4}{y_4}$ of the two arbitrary points 3 and 4 on the circle) the equation must become an identity in regard to the terms in dx^3 and dx_4 respectively. It only remains to verify that this is so.

14. Writing $P, Q, R = -y_3 + y_4, \; x_3 - x_4, \; x_3y_4 - x_4y_3$; also L_3 and L_4 $= x_3 + y_3 - 1$ and $x_4 + y_4 - 1$ respectively, we have $a = P \div R$, $b = Q \div R$, and the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{1}{(Q-R)(R-P)}\{(Q-R)\,dP + (R-P)\,dQ + (P-Q)\,dR\},$$

where, substituting for dy_3, dy_4 their values in terms of dx_3, dy_3, we have

$$dP,\ dQ,\ dR = \frac{1}{y_3}\,x_3\,dx_3 - \frac{1}{y_4}\,x_4\,dx_4,\ \ \frac{1}{y_3}\,y_3\,dx_3 - \frac{1}{y}\,y_4\,dx_4,$$
$$\frac{1}{y_3}\,(x_3x_4 + y_3y_4)\,dx_3 - \frac{1}{y_4}\,(x_3x_4 + y_3y_4)\,dx_4,$$

and with these values, and by aid of the relations $Q - R,\ R - P,\ P - Q$ $= x_4L_3 - x_3L_4,\ y_4L_3 - y_3L_4,\ -L_3 + L_4$, the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{L_3L_4(x_3x_4 + y_3y_4 - 1)}{(x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4)}\left(\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4}\right);$$

viz. this will be true if only

$$L_3L_4(x_3x_4 + y_3y_4 - 1) - (x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4) = 0,$$

that is

$$-x_4y_4L_3^2 - x_3y_3L_4^2 + L_3L_4(x_3x_4 + y_3y_4 + x_3y_4 + x_4y_3 - 1) = 0.$$

But from the values of L_3, L_4 we have $x_4y_4 = \frac{1}{2}L_4^2 + L_4$, $x_3y_3 = \frac{1}{2}L_3^2 + L_3$, and the coefficient of L_3L_4 is $= L_3L_4 + L_3 + L_4$; the equation is thus verified.

The example would perhaps have been more instructive if the points 1 and 2 had been left arbitrary points on the circle, but the working out would have been more difficult.

The Variable Intersections of the Two Curves—Number of Independent Integral Relations. Art. Nos. 15 to 19.

15. Suppose $n = 3$, $\delta = 0$ $(p = 1)$, the fixed curve a cubic; and suppose successively $m = 1, 2, 3, \ldots$ the variable curve a line, conic, cubic, etc.

If $m = 1$, then two points on the cubic determine the line, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m = 2$, then five points on the cubic determine the conic, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m = 3$, then eight points on the fixed cubic do not determine the variable cubic, but they do determine the ninth intersection. For draw through the eight points a no-matter-what cubic $\chi = 0$, the general cubic through the eight

points is $\chi + \alpha f = 0$, and this meets the fixed cubic in the points $\chi = 0$, $f = 0$, that is in the eight points and in one other point independent of the constant α and therefore completely determinate. Hence in this case also there is one integral relation.

So if $m = 4$, then eleven points on the cubic do not determine the quartic, but they do determine the remaining intersection. For draw through the eleven points a no-matter-what quartic $\chi = 0$, the general quartic through the eleven points is $\chi + (x, y, z)^1 f = 0$, and this meets the cubic in the points $\chi = 0, f = 0$, that is in the eleven points and in one other point independent of the constants of the linear function $(x, y, z)^1$, and therefore completely determinate. Hence there is one integral relation.

And so in general, the fixed curve being a cubic, then whatever be the order of the variable curve, there is always one integral relation.

16. Suppose next $n = 4$, $\delta = 0\,(p = 3)$, the fixed curve a general quartic; and as before suppose successively $m = 1, 2, 3, \ldots$ the variable curve a line, conic, cubic, etc.

If $m = 1$, then two points on the quartic determine the line, and therefore the remaining two intersections; the number of integral relations is $= 2$.

If $m = 2$, then five points on the quartic determine the conic, and therefore the remaining three intersections; the number of integral relations is $= 3$, and similarly if $m = 3$, the number of integral relations is $= 3$.

If $m = 4$, then thirteen points on the fixed quartic do not determine the variable quartic, but they do determine the remaining three intersections. For draw through the thirteen points a no-matter-what quartic $\chi = 0$; the general quartic through the thirteen points is $\chi + \alpha f = 0$, and this meets the fixed quartic in the points $\chi = 0$, $f = 0$, that is in the thirteen points and in three other points, independent of α and thus completely determinate, and the number of integral relations is $= 0$; and so in general for any higher value of m, the number is still $= 3$.

17. Suppose $m = 5$, $\delta = 0\,(p = 6)$, the fixed curve a general quintic, and as before $m = 1, 2, 3 \ldots$ successively.

If $m = 1$, then two points on the quintic determine the line, and therefore the remaining three intersections; the number of integral relations is $= 3$.

If $m = 2$, then five points on the quintic determine the conic, and therefore the remaining five intersections; the number of integral relations is $= 5$.

If $m = 3$, then 9 points on the quintic determine the cubic, and therefore the remaining six intersections; the number of integral relations is $= 6$, and so if $m = 4$, or if $m = 5$ or any greater number, in the first case directly, and in the other cases by consideration of the quintic $\chi + \alpha f = 0$, etc., we find that the number of integral relations is always $= 6$.

18. The reasoning is scarcely altered in the case where the fixed curve has dps; thus considering the general case of a fixed curve n, δ, p:

If $m = 1$, then $2 - \delta$ points on the fixed curve (this implies $\delta \not> 2$) determine the line, and therefore the remaining $n - 2\delta - (2 - \delta)$, $= n - 2 - \delta$ intersections; the number of integral relations is $= n - 2 - \delta$.

If $m = 2$, then $5 - \delta$ points on the fixed curve (this implies $\delta \not> 5$) determine the conic, and therefore the remaining $2n - 2\delta - (5 - \delta)$, $= 2n - 5 - \delta$ intersections; the number of integral relations is $= 2n - 5 - \delta$, and so for any value of $m \not> n - 3$, and indeed for the values $n - 2$ and $n - 1$; here $\frac{1}{2} m(m + 3) - \delta$ points on the fixed curve (this implies $\delta \not> \frac{1}{2} m(m + 3)$) determine the variable curve, and therefore the remaining $mn - 2\delta - (\frac{1}{2} m(m + 3) - \delta)$, $= mn - \frac{1}{2} m(m + 3) - \delta$ intersections. Hence the number of integral relations is $= mn - \frac{1}{2} m(m + 3) - \delta$, that is, $= p - \frac{1}{2}(n - m - 1)(n - m - 2)$. And thus in the cases $m = n - 2$ and $n - 1$ the number is $= p$.

If $m = n$, then $\frac{1}{2} n(n + 3) - 1 - \delta$ points on the fixed curve do not determine the variable curve, but they do determine the remaining $n^2 - 2\delta - (\frac{1}{2} n(n + 3) - 1 - \delta)$, $= \frac{1}{2}(n - 1)(n - 2) - \delta$, that is p intersections, and the number of integral relations is thus $= p$; and so for any higher value of m the number is still $= p$.

19. The conclusion thus is

$m \not> n - 3$ number of integral relations $= p - \frac{1}{2}(n - m - 1)(n - m - 2)$,
$m =$ or $> n - 2$ " " $= p$.

The integral equations spoken of throughout are of course independent relations.

The Variable Intersections of the Two Curves. Number of Independent Displacement Relations given by the Pure Theorem. Art. No. 20.

20. The number of displacement-relations given by the pure theorem is = number of constants in minor function $(x, y, z)^{n-3}$, which equated to zero represents a curve through the dps, viz. this is always

$$\tfrac{1}{2}(n - 1)(n - 2) - \delta, = p.$$

But for $m > n-2$, these relations are not independent. For instance, for $n=4$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^1 d\omega = 0, \text{ that is } \Sigma x d\omega = 0,\ \Sigma y d\omega = 0,\ \Sigma z d\omega = 0,$$

and conversely from these last equations we have $\Sigma(x, y, z)^1 d\omega = 0$. But in this case the variable curve is a line $ax + by + cz = 0$; hence writing $(x, y, z)^1 = ax + by + cz$, the equation $(x, y, z)^1 = 0$ is satisfied for each of the intersections of the line with the quartic, and the corresponding equation $\Sigma(x, y, z)^1 d\omega = 0$ is an identity. Hence the number of independent displacement-relations is $3-1, =2$.

So for $n=5$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^2 d\omega = 0, \text{ that is } \Sigma(x^2, y^2, z^2, yz, zx, xy)\, d\omega = 0,$$

and these six equations give conversely $\Sigma(x, y, z)^2 d\omega = 0$, and in particular they give $\Sigma x(x, y, z)^1 d\omega = 0$, $\Sigma y(x, y, z)^1 d\omega = 0$, $\Sigma z(x, y, z)^1 d\omega = 0$. But if $(x, y, z)^1$ denote $ax + by + cz$, then as before we have $(x, y, z)^1 = 0$, for each of the intersections of the two curves, and the last mentioned three equations are identities. The number of independent displacement-relations is thus $6-3, =3$.

So for $n=5$, $\delta=0$, $m=2$. Here if the variable curve is $\phi = (a, \ldots \between x, y, z)^2 = 0$, then taking $(x, y, z)^2 = (a, \ldots \between x, y, z)^2$, the equation $(x, y, z)^2 = 0$ is satisfied for each of the intersections of the two curves, and the corresponding equation $\Sigma(x, y, z)^2 d\omega = 0$ is an identity; the number of independent displacement-relations is $6-1, =5$.

The reasoning is the same when δ is not $=0$, and we see generally that for $m < n-2$, or say

$$m \not> n-3, \text{ number of independent displacement-relations}$$
$$= p - \tfrac{1}{2}(n-m-1)(n-m-2);$$

while for $m =$ or $> n-2$, number is $=p$;

since in this case the relations given by the theorem are all of them independent. It thus appears *à posteriori*, that in every case the number of independent displacement-relations given by the pure theorem is equal to the number of independent integral relations.

As to the dps of the Fixed Curve. Art. No. 21.

I conclude with a general remark applicable to the whole of the three chapters. There is no necessity to attend to all or indeed to any of the dps of the fixed curve. Suppose that the fixed curve has $\delta + \delta'$ dps, where δ, δ' may

be either of them or each $=0$, but attend only to the δ dps, the δ' dps being wholly disregarded, and accordingly let the expression the dps mean as before the δ dps of the fixed curve. No alteration at all is required, only if δ' be not $=0$, then $p=\frac{1}{2}(n-1)(n-2)-\delta$ will no longer be the deficiency. To obtain the best theorems we use all the $\delta+\delta'$ dps, but disregarding the δ' dps, we obtain theorems, as for a curve with δ dps, which are true, and may frequently be useful either in their original form or with simplifications introduced therein by afterwards taking account of the δ' dps.

CHAPTER II. PROOF OF ABEL'S THEOREM.

Preparation. Art. Nos. 22 and 23.

22. Starting from the equation $\phi=(\mathrm{a},\ldots\between x, y, z)^m=0$ of the variable curve, we have

$$\frac{d\varphi}{dx}dx+\frac{d\varphi}{dy}dy+\frac{d\varphi}{dz}dz+\delta\phi=0,$$

$$\frac{d\varphi}{dx}x+\frac{d\varphi}{dy}y+\frac{d\varphi}{dz}z=0,$$

where $\delta\phi=(d\mathrm{a},\ldots\between x, y, z)^m$. Let τ denote an arbitrary linear function, $=ax+by+cz$, multiply the two equations by $ax+by+cz$, $=\tau$, and $-(adx+bdy+cdz)$, $=-d\tau$ respectively, and add, we obtain

$$(ydz-zdy)\left(b\frac{d\varphi}{dz}-c\frac{d\varphi}{dy}\right)+(zdx-xdz)\left(c\frac{d\varphi}{dx}-a\frac{d\varphi}{dz}\right)+(xdy-ydx)\left(a\frac{d\varphi}{dy}-b\frac{d\varphi}{dx}\right)+\tau\delta\phi=0;$$

introducing $d\omega$, this becomes

$$d\omega\left[\frac{df}{dx}\left(b\frac{d\varphi}{dz}-c\frac{d\phi}{dy}\right)+\frac{df}{dy}\left(c\frac{d\varphi}{dx}-a\frac{d\varphi}{dz}\right)+\frac{df}{dz}\left(a\frac{d\varphi}{dy}-b\frac{d\varphi}{dx}\right)\right]+\tau\delta\phi=0,$$

or observing that a, b, c are the differential coefficients $\frac{d\tau}{dx}$, $\frac{d\tau}{dy}$, $\frac{d\tau}{dz}$, the term in [ ] is $J(f,\tau,\phi)$, or say $J(\phi,f,\tau)$, and the equation is

$$d\omega J(\phi,f,\tau)+\tau\delta\phi=0,$$

where $J(\phi,f,\tau)$ is the Jacobian, or functional determinant

$$\begin{vmatrix} \frac{d\varphi}{dx}, & \frac{d\varphi}{dy}, & \frac{d\varphi}{dz} \\ \frac{df}{dx}, & \frac{df}{dy}, & \frac{df}{dz} \\ \frac{d\tau}{dx}, & \frac{d\tau}{dy}, & \frac{d\tau}{dz} \end{vmatrix},=\frac{d(\varphi,f,\tau)}{d(x,y,z)};$$

and we hence have

$$d\omega = \frac{-\tau\delta\varphi}{J(\varphi, f, \tau)}.$$

23. The two theorems thus become

$$\Sigma\,(x,\,y,\,z)^{n-3}\frac{\tau\delta\varphi}{J(\varphi, f, \tau)} = 0\,,$$

$$\Sigma\frac{(x,\,y,\,z)_{12}^{n-2}}{012}\cdot\frac{-\tau\delta\varphi}{J(\varphi, f, \tau)} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}.$$

But further, if in the first equation we write $\tau = z$, and in the second equation we retain τ, using it to denote the linear function 012, the equations become

$$\Sigma\,(x,\,y,\,z)^{n-3}\frac{z\delta\varphi}{J(\varphi, f)} = 0\,;$$

$$\Sigma\,(x,\,y,\,z)_{12}^{n-2}\cdot\frac{-\delta\varphi}{J(\varphi, f, \tau)} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\,;$$

where in the first equation $J(\phi,\,f)$ denotes the Jacobian

$$\begin{vmatrix} \dfrac{d\varphi}{dx}, & \dfrac{d\varphi}{dy} \\ \dfrac{df}{dx}, & \dfrac{df}{dy} \end{vmatrix}, = \frac{d(\varphi, f)}{d(x,\,y)}.$$

In these equations the only differential symbol is the δ affecting the coefficients a, b, ... of ϕ, ϕ_1, ϕ_2; the equations are in respect to the coordinates $(x,\,y,\,z)$ of the several variable intersections of the two curves, purely algebraical equations, which are in fact given by Jacobi's Fraction-theorem about to be explained. But for the further reduction of the affected theorem I interpose the next article.

The Coordinates $(\rho,\,\sigma,\,\tau)$. Art. Nos. 24 to 26.

24. The letter τ has just been used to denote the determinant 012: there is often occasion to consider three points 1, 2, 3 coordinates $(x_1,\,y_1,\,z_1)$, $(x_2,\,y_2,\,z_2)$, $(x_3,\,y_3,\,z_3)$ respectively; and then writing ρ, σ, τ to denote the determinants 023, 031, 012 respectively, and Δ the determinant 123, we have identically

$$\Delta x = x_1\rho + x_2\sigma + x_3\tau,$$
$$\Delta y = y_1\rho + y_2\sigma + y_3\tau,$$
$$\Delta z = z_1\rho + z_2\sigma + z_3\tau,$$

which equations, regarding therein the point 0, coordinates $(x,\,y,\,z)$, as a current point, are in fact equations for transformation from the coordinates $(x,\,y,\,z)$ to the coordinates ρ, σ, τ belonging to the triangle of reference 123. The points

1 and 2 have been already taken to be on the fixed curve, and it will be assumed that 3 is also a point on this curve.

25. If the function f, which equated to zero gives the equation of the fixed curve, be transformed to the new coordinates (ρ, σ, τ), the coefficients of the transformed function are polar functions, each divided by ∇^n, viz. the coefficient of ρ^n is $\frac{1}{\nabla^n} 1^n$, which by reason that 1 is a point on the curve is $= 0$ (and similarly the coefficients of σ^n and of τ^n are each $= 0$), the coefficient of $\rho^{n-1}\sigma$ is $= \frac{1}{\nabla^n} n.1^{n-1}2$; that of $\rho^{n-1}\tau$ is $= \frac{1}{\nabla^n} n.1^{n-1}3$; that of $\rho^{n-2}\sigma^2$ is $= \frac{1}{\nabla^n} \frac{1}{2} n(n-1) 1^{n-2}2^2$; and so in other cases. I write this in the form

$$f = \frac{1}{\nabla^n}(1^n = 0, \ldots)(\rho, \sigma, \tau)^n;$$

or we might also use the symbolic form

$$f = \frac{1}{\Delta^n}(\rho 1 + \sigma 2 + \tau 3)^n.$$

The terms independent of τ contain, it is clear, the factor $\rho\sigma$, and separating these terms from the others, the equation may be written

$$f = \frac{1}{\nabla^n}\rho\sigma(n.1^{n-1}2, \ldots \dagger)(\rho, \sigma)^{n-2} + \&\text{c.}\,\tau.$$

26. The equations $\tau = 0$, $(\ldots \dagger)(\rho, \sigma)^{n-2} = 0$, determine the residues of the points 1, 2, and hence the major function $(x, y, z)_{12}^{n-2}$, expressed in terms of ρ, σ, τ, and writing therein $\tau = 0$, must reduce itself to $(\ldots \dagger)(\rho, \sigma)^{n-2}$ into a constant factor which is at once found to be $= \frac{1}{\nabla^{n-2}}$; for taking the current point at 1 we have $(\rho, \sigma, \tau) = (\Delta, 0, 0)$, and the corresponding value of the major function is thus $\frac{1}{\nabla^{n-2}} n.1^{n-1}2.\Delta^{n-2}$, $= n.1^{n-2}2$, as it ought to be. We have thus

$$(x, y, z)_{12}^{n-2} = \frac{1}{\nabla^{n-2}}(n.1^{n-1}2, \ldots \dagger)(\rho, \sigma)^{n-2} + \&\text{c.}\,\tau;$$

and we hence see that for $\tau = 0$,

$$(x, y, z)_{12}^{n-2} = \frac{\Delta^2 f}{\rho\sigma},$$

an equation which will be useful.

The Preparation for the Affected Theorem Resumed. Art. No. 27.

27. In the affected theorem instead of (x, y, z) we introduce the new coordinates (ρ, σ, τ). We have

$$J(\phi, f, \tau) = \frac{d(\varphi, f, \tau)}{d(\rho, \sigma, \tau)} \frac{d(\rho, \sigma, \tau)}{d(x, y, z)},$$

where the first factor is $=\dfrac{d(\varphi, f)}{d(\rho, \sigma)}$, say that this is $\overline{J}(\phi, f)$ the Jacobian in regard to ρ, σ, and the second factor is at once found to be $=\Delta^2$. We have consequently

$$\frac{1}{J(\varphi, f, \tau)}=\frac{1}{\Delta^2\overline{J}(\varphi, f)},$$

and the equation for the affected theorem becomes

$$\Sigma\,(x, y, z)_{12}^{n-2}\frac{\delta\varphi}{\overline{J}(\varphi, f)}=-\Delta^2\Big(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\Big),$$

where $(x, y, z)_{12}^{n-2}$ is to be regarded as standing for its value in terms of (ρ, σ, τ).

Jacobi's Fraction Theorem. Art. Nos. 28 to 31.

28. This is the extension of a well-known theorem, which, in a somewhat disguised form, may be thus written: viz. if U be any rational and integral function $(x, 1)^m$, then we have

$$\frac{1}{U}=\Sigma\frac{1}{x-x'.J(U')},$$

or introducing an arbitrary constant A by the equation $AU=X$, say this is

$$\frac{A}{X}\Big(=\frac{1}{U}\Big)=\Sigma\frac{1}{x-x'.J(U')},$$

where U' is the same function $(x', 1)^m$ of x' that U is of x: $J(U'), =\dfrac{dU'}{dx'}$ is the Jacobian of U', and the summation extends to all roots x' of the equation $U'=0$: obviously this is nothing else than the formula for the decomposition of $\dfrac{1}{U}$ into simple fractions.

29. Take now $U=(x, y, 1)^m$, $V=(x, y, 1)^n$, functions of x, y of the degrees m and n respectively, and assume

$$\begin{aligned} AU+BV&=X, \text{ a function } (x, 1)^{mn}, \\ CU+DV&=Y, \quad\text{"}\quad (y, 1)^{mn}, \end{aligned}$$

viz. let $X=0$ and $Y=0$ be the equations obtained by elimination from $U=0$ and $V=0$ of the y and the x respectively. The forms are

$$\begin{aligned} A&=(x, y^{n-1}, 1)^{mn-m}, \quad & B&=(x, y^{m-1}, 1)^{mn-n}, \\ C&=(x^{n-1}, y, 1)^{mn-m}, \quad & D&=(x^{m-1}, y, 1)^{mn-n}, \end{aligned}$$

where these equations denote the first of them that A is a rational and integral function of the degree $mn-m$ in x and y jointly, but only of the degree $n-1$ in y: and so for the other equations. It follows that

$$AD-BC=(x^{mn-1}, y^{mn-1}, 1)^{2mn-m-n}.$$

The theorem now is

$$\frac{AD-BC}{XY}=\Sigma\frac{1}{x-x'.y-y'.J(U',\ V')},$$

where U', V' are the same functions of (x', y') that U, V are of (x, y); $J(U', V')$ is the Jacobian $\frac{d(U',\ V')}{d(x',\ y')}$; and the summation extends to all the simultaneous roots x', y' of the equations $U=0$, $V=0$.

30. For the proof, observe that $AD-BC$ is a sum of terms of the form $x^\alpha y^\beta$ where α and β are each of them at most $=mn-1$; hence X being of the degree mn we have $\frac{x^\alpha}{X}=$ a sum of fractions $\frac{L}{x-x'}$, where x' is any root of $X=0$; and similarly $\frac{y^\beta}{Y}=$ a sum of fractions $\frac{M}{y-y'}$, where y' is any root of $Y=0$; multiplying the two expressions and taking the sum for the several terms $\lambda x^\alpha y^\beta$ of $AD-BC$ we obtain a formula

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'.y-y'},$$

where the summation extends to all the combinations of the mn values of x' with the mn values of y. But such a formula existing, the coefficients K may be determined in the usual manner, viz. multiplying by XY and then writing $x=x'$, $y=y'$, there is on the right-hand only one term which does not vanish, and we find

$$(AD-BC)_{x'y'}=K\left(\frac{X}{x-x'}\right)_{x'}\left(\frac{Y}{y-y'}\right)_{y'},=K\left(\frac{dX}{dx}\,\frac{dY}{dy}\right)_{x'y'},$$

where the factor which multiplies K does not vanish.

We distinguish the cases where (x', y') are corresponding or non-corresponding roots of $X=0$, $Y=0$; viz. corresponding roots are those for which $U=0$, $V=0$, but for non-corresponding roots these equations do not hold good; there are obviously mn pairs of corresponding roots.

In the latter case $(AD-BC)\,U=DX-BY$; $(AD-BC)\,V=-CX+AY$, and since for the values in question X, Y each vanish, but U, V do not each of them vanish, we must for these values have $AD-BC=0$, and the foregoing equation for K gives then $K=0$.

31. The formula thus is

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'.y-y'},$$

where the summation now extends only to corresponding roots x', y', for which we have $U=0$, $V=0$. We have for K the foregoing expression, which, to complete the determination, we write under the form

$$AD-BC=KJ(X,\ Y)_{x'y'};$$

this is allowable, for $J(X,\ Y),=\dfrac{d(X,\ Y)}{d(x,\ y)}$, differs from $\dfrac{dX}{dx}\dfrac{dY}{dy}$ only by the zero term $-\dfrac{dX}{dy}\dfrac{dY}{dx}$. Moreover, differentiating the expressions for X, Y, and considering $(x,\ y)$ as therein standing for a pair of corresponding roots $(x',\ y')$, the terms containing U, V will all vanish; we thus in effect differentiate as if A, B, C, D were constants, and the result is $(AD-BC)\,J(U,\ V)$, or say this is $(AD-BC)_{x'y'}\,J(U',\ V')$: hence, in the equation for K, the factor $(AD-BC)_{x'y'}$ divides out, and we have $1=KJ(U',\ V')$; hence the required formula is

$$\frac{AD-BC}{XY}=\Sigma\,\frac{1}{x-x'.\,y-y'}\,.\,J(U',\ V')$$

the summation extending to all the simultaneous roots $(x',\ y')$ of $U=0$, $V=0$.

Homogeneous Form of the Fraction Theorem. Art. Nos. 32 and 33.

32. For x, y, x', y' we write $\dfrac{x}{z}$, $\dfrac{y}{z}$, $\dfrac{x'}{z'}$, $\dfrac{y'}{z'}$; supposing that U, V now denote homogeneous functions $(x,\ y,\ z)^m$, $(x,\ y,\ z)^n$, and that we have

$$\begin{aligned} AU+BV&=X,\ =(x,\ z)^{mn},\ =\alpha x^{mn}+\ldots\\ CU+DV&=Y,\ =(y,\ z)^{mn},\ =\beta y^{mn}+\ldots \end{aligned}$$

where the forms are

$$\begin{gathered} A=(x,\ y^{n-1},\ z)^{mn-m},\quad B=(x,\ y^{m-1},\ z)^{mn-n},\\ C=(x^{n-1},\ y,\ z)^{mn-m},\quad D=(x^{m-1},\ y,\ z)^{mn-n},\\ AD-BC=(x^{mn-1},\ y^{mn-1},\ z)^{2mn-m-n}, \end{gathered}$$

(viz. the degree of A in $(x,\ y,\ z)$ is $=mn-m$, but y rises only to the degree $n-1$; and so in other cases); then the theorem becomes

$$\frac{z^{m+n-2}\,(AD-BC)}{XY}=\Sigma\,\frac{z'^{m+n}}{xz'-x'z\,.\,yz'-y'z}\,.\,J(U',\ V'),$$

where $J(U',\ V')$ denotes the Jacobian $\dfrac{d(U',\ V')}{d(x',\ y')}$, and the summation extends to the simultaneous roots $(x',\ y',\ z')$ of $U=0$, $V=0$.

32. It is proper to introduce into the formula τ', an arbitrary linear function $ax'+by'+cz'$ of (x', y', z'): observe that in the Jacobian, (x', y', z') have always values for which $U'=0$, $V'=0$: we have therefore

$$x'\frac{dU'}{dx'}+y'\frac{dU'}{dy'}+z'\frac{dU'}{dz'}=0,$$
$$x'\frac{dV'}{dx'}+y'\frac{dV'}{dy'}+z'\frac{dV'}{dz'}=0,$$

and thence

$$x':y':z'=\frac{d(U', V')}{d(y', z')}:\frac{d(U', V')}{d(z', x')}:\frac{d(U', V')}{d(x', y')},$$

and if the expressions on the right-hand are for a moment called A', B', C', then writing $\tau'=ax'+by'+cz'$, we have $J(U', V', \tau')=aA'+bB'+cC'$, $=\frac{\tau'}{z'}C'$, $=\frac{\tau'}{z'}J(U', V')$, that is $\frac{1}{J(U', V')}=\frac{\tau'}{z'J(U', V', \tau')}$, or the equation becomes

$$\frac{z^{m+n-2}(AD-BC)}{XY}=\Sigma\frac{z'^{m+n-1}\tau'}{xz'-x'z.yz'-y'z}\cdot J(U', V', \tau'),$$

the summation being as before.

Resulting Special Theorems. Art. Nos. 33–35

33. Reverting to the Cartesian form, we have

$$\frac{xy(AD-BC)}{XY}=\Sigma\frac{1}{J(U', V')}\Big(1+\frac{x'}{x}+\ldots\Big)\Big(1+\frac{y'}{y}+\ldots\Big),$$
$$=\Sigma\frac{1}{J(U', V')}\Big\{1+H_1\Big(\frac{x'}{x}, \frac{y'}{y}\Big)+H_2\Big(\frac{x'}{x}, \frac{y'}{y}\Big)+\ldots\Big\}$$

where H_m is the homogeneous sum of the order m, $H_1(u, v)=u+v$, $H_2(u, v)=u^2+uv+v^2$, &c.

The left-hand side is

$$(AD-BC)\Big(\frac{1}{\alpha x^{mn-1}}+\frac{\&c.}{x^{mn}}\ldots\Big)\Big(\frac{1}{\beta y^{mn-1}}+\frac{\&c.}{y^{mn}}+\ldots\Big)$$

and in $AD-BC$ the terms of highest order in (x, y), say $(AD-BC)_0$ are $(AD-BC)_0=(xy)^{mn-m-n+1}(a, b\ldots k\,\dagger\!\!\between x, y)^{m+n-2}$.

There is thus on the left-hand no term which is in (x, y) of a higher degree than $-(m+n-2)$; hence on the right-hand every term of a higher degree than this in (x, y) must vanish, viz. we must have

$$0=\Sigma\frac{x'^{\alpha}y'^{\beta}}{J(U', V')}\text{ so long as }\alpha+\beta \not> m+n-3,$$

or what is the same thing, we must have

$$0 = \Sigma \frac{(x', y', 1)^{m+n-3}}{J(U', V')} \qquad (m+n-3) \text{ theorem.}$$

where $(x', y', 1)^{m+n-3}$ is the arbitrary function of the degree $m+n-3$.

34. Passing to the next lower degree $-(m+n-2)$ we have

$$\frac{1}{\alpha\beta(xy)^{m+n-2}}(a, b, \ldots k \between x, y)^{m+n-2} = \Sigma \frac{1}{J(U', V')} H_{m+n-2}\left(\frac{x'}{x}, \frac{y'}{y}\right)$$

and if in $(a, b, \ldots k \between x, y)^{m+n-2}$ we consider any term $gx^{m+n-2-p}y^{m+n-2-q}$, where $p+q = m+n-2$, then we have on the left-hand the term $\frac{g}{\alpha\beta x^p y^q}$, and the corresponding term on the right-hand must be $\Sigma \frac{1}{J(U', V')} \frac{x'^p y'^q}{x^p y^q}$; that is we have

$$\frac{g}{\alpha\beta} = \Sigma \frac{x'^p y'^q}{J(U', V')}.$$

But from the foregoing expression for $(AD-BC)_0$ it appears that $(AD-BC)_0$ contains the term $gx^{mn-1-p}y^{mn-1-q}$, and it hence appears that g is the constant term of the quotient $(AD-BC)_0$ divided by $x^{mn-1-p}y^{mn-1-q}$, or as this may be written

$$g = \text{const. of } \frac{(AD-BC)_0 x^p y^q}{(xy)^{mn-1}}$$

and comparing the two values of g we obtain

$$\text{const. of } \frac{(AD-BC)_0 x^p y^q}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{x'^p y'^q}{J(U', V')}, \quad (p+q = m+n-2),$$

and we hence derive

$$\text{Const. of } \frac{(AD-BC)_0 (x, y)^{m+n-2}}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{(x', y')^{m+n-2}}{J(U', V')},$$

where $(x, y)^{m+n-2}$ is the general function of the degree $m+n-2$, and, of course, $(x', y')^{m+n-2}$ is the same function of x', y'. The two functions may be written $(x, y, 0)^{m+n-2}$ and $(x', y', 0)^{m+n-2}$, and this being so we may on the right-hand write instead $(x', y', 1)^{m+n-2}$, for, by so doing we introduce in the numerator of the fraction new terms of an order not exceeding $m+n-3$, and by the $(m+n-3)$ theorem already obtained the sum Σ of the quotient of such terms by $J(U', V')$ is $= 0$. We thus have

$$\text{Const. of } \frac{(AD-BC)_0 (x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}} = \Sigma \frac{(x', y', 1)^{m+n-2}}{J(U', V')}, \quad (m+n-2) \text{ theorem.}$$

where $(x', y', 1)^{m+n-2}$ is the general non-homogeneous function of the degree $m+n-2$, and $(x, y, 0)^{m+n-2}$ is obtained from it by attending only to the terms of the highest degree $m+n-2$, and therein substituting x, y for x', y'.

35. We may, it is clear, in the equations for the $(m+n-3)$ and for the $(m+n-2)$ theorems respectively, omit the accents on the right-hand sides; doing this, and moreover in each equation transposing the two sides, the two special theorems are

$$\Sigma\frac{(x,\,y,\,1)^{m+n-3}}{J(U,\,V)}=0,\qquad (m+n-3)\text{ theorem.}$$

$$\Sigma\frac{(x,\,y,\,1)^{m+n-2}}{J(U,\,V)}=\text{const. of }\frac{(AD-BC)_0(x,\,y,\,0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}.\qquad (m+n-2)\text{ theorem.}$$

Homogeneous Form of the Special Theorems. Art. No. 36.

36. Writing $\frac{x}{z},\ \frac{y}{z}$ for $x,\,y$, and introducing as before the arbitrary linear function $\tau=ax+by+cz$, we at once obtain, $U,\,V$ being now homogeneous functions $(x,\,y,\,z)^m$ and $(x,\,y,\,z)^n$ respectively, and the $A,\,B,\,C,\,D$ being also homogeneous functions accordingly,

$$\Sigma\frac{z(x,\,y,\,z)^{m+n-3}}{J(U,\,V)}=0,\qquad (m+n-3)\text{ theorem.}$$

$$\Sigma\frac{\tau(x,\,y,\,z)^{m+n-2}}{zJ(U,\,V,\,\tau)}=\text{const. of }\frac{(AD-BC)_0(x,\,y,\,0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}},\quad (m+n-2)\text{ theorem.}$$

where the suffix 0 denotes that we are in $AD-BC$ to write $z=0$.

If in the last formula we change throughout the letters $x,\,y,\,z$ into $\rho,\,\sigma,\,\tau$ (that is, consider $U,\ V$ as given functions of $\rho,\,\sigma,\,\tau$), but retain τ as standing for the particular function $0\rho+0\sigma+1\tau$, then the formula becomes

$$\Sigma\frac{(\rho,\,\sigma,\,\tau)^{m+n-2}}{\overline{J}(U,\,V)}=\text{const. of }\frac{(AD-BC)_0(\rho,\,\sigma,\,\tau)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}},\quad (m+n-2)\text{ theorem.}$$

where $\overline{J}(U,\,V)$ denotes $\frac{d(U,\,V)}{d(\rho,\,\sigma)}$, the Jacobian in regard to $\rho,\,\sigma$.

The effect of dps of the Curves $U=0,\ V=0$. Art. Nos. 37 and 38.

37. We must, in regard to the foregoing special theorems, consider the effect of any dps of the curves $U=0,\ V=0$.

Suppose one of the curves, say V, has a dp, but that the other curve U does not pass through it; the dp is not an intersection of $U,\ V$, and the theorems are in nowise affected.

If U passes through the dp then the dp counts twice among the intersections of $U,\ V$; at the dp we have $J(U',\ V')=0$, and (to fix the ideas attending to the $(m+n-3)$ theorem) the sum $\Sigma\frac{(x,\,y,\,z)^{m+n-3}}{J(U,\,V)}$ will contain two

infinite terms; these may very well, and indeed (assuming that the theorem remains true) must have a finite sum, but except by the theorem itself, this finite sum is not calculable, and the theorem thus becomes nugatory.

If, however, the curve $(x, y, z)^{m+n-3}=0$ be a curve passing through the dp, then considering, instead, the case where the last-mentioned curve and U each approach indefinitely near to the dp of V; there are two intersections of U, V indefinitely near to each other and to the dp; at either intersection, the numerator $(x', y', z')^{m+n-3}$ and the denominator $J(U, V)$ are infinitesimals of the same order, say the first, and the fraction has a finite value; the finite values for the two intersections have not in general a zero sum, and consequently in the limit it would not be allowable to disregard the intersections belonging to the dp.

38. But if the numerator curve $(x, y, z)^{m+n-3}=0$ passes twice through the dp (that is, has there a dp), then reverting to the two consecutive intersections, at either of these the denominator $J(U, V)$ is as before an infinitesimal of the first order, but the numerator $(x, y, z)^{m+n-3}$ is an infinitesimal of the second order, and in the limit the value of the fraction is $=0$; we may in this case disregard the intersections belonging to the dp; and so in general, the curve $(x, y, z)^{m+n-3}=0$ passing twice through each dp of U which lies upon V, we have

$$\Sigma \frac{z(x, y, z)^{m+n-3}}{J(U, V)}=0,$$

the summation now extending to all the intersections of U, V other than the dps in question, which are to be disregarded. And the like in regard to the other theorem

$$\Sigma \frac{(x, y, z)^{m+n-2}}{J(U, V)}=\text{const. of } \frac{(AD-BC)_0 (x, y, 0)^{m+n-2}}{\alpha\beta\,(xy)^{mn-1}}.$$

The Pure Theorem.—Completion of the Proof. Art. No. 39.

39. The theorem was reduced to

$$\Sigma \frac{z(x, y, z)^{n-3}\delta\varphi}{J(\varphi, f)}=0,$$

which is therefore the equation to be proved.

The $(m+n-3)$ theorem, writing therein ϕ, f in place of U, V respectively (the degrees being as before m and n), is

$$\Sigma \frac{z(x, y, z)^{m+n-3}}{J(U, V)}=0.$$

$(x, y, z)^{m+n-3}$ is here an arbitrary function of the degree $m+n-3$, and this may therefore be put $=(x, y, z)^{n-3}\delta\phi$, where $\delta\phi=(da, \ldots \between x, y, z)^m$, is a function of the degree m; and since the curve $\phi=0$ passes always through the dps of f, and varies subject to this condition, the curve $\delta\phi=0$ will also pass through the dps; hence taking $(x, y, z)^{n-3}=0$ a curve through the dps, the curve $(x, y, z)^{n-3}\delta\phi=0$ will be a curve passing twice through each of dps, and the $(m+n-3)$ theorem thus gives the equation which was to be proved. This completes the proof of the pure theorem

$$\Sigma\,(x, y, z)^{n-3}d\omega=0.$$

The Affected Theorem.—Completion of the Proof. Art. Nos. 40 and 41.

40. The theorem was reduced to

$$\Sigma\,\frac{(x, y, z)_{12}^{n-2}\,\delta\varphi}{J(\varphi, f)}=-\Delta^2\Big(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\Big),$$

which is therefore the equation to be proved.

The $(m+n-2)$ theorem, written with (ρ, σ, τ) in place of (x, y, z), and putting therein ϕ, f for U, V, is

$$\Sigma\,\frac{(\rho, \sigma, \tau)^{m+n-2}}{J(\varphi, f)}=\text{const. of }\frac{(AD-BC)_0(\rho, \sigma, 0)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where it will be recollected that the suffix (0) denotes that τ is to be put $=0$. $(\rho, \sigma, \tau)^{m+n-2}$ is here an arbitrary function of the degree $m+n-2$, and this may therefore be put $=(x, y, z)_{12}^{n-2}\delta\phi$, the two factors being each of them considered as expressed in terms of (ρ, σ, τ); and since each of the curves $(x, y, z)_{12}^{n-2}=0$ and $\delta\phi=0$ passes through the dps of f, the curve $(x, y, z)_{12}^{n-2}\delta\phi=0$, is a curve passing twice through each of the dps. We have therefore

$$\Sigma\,\frac{(x, y, z)_{12}^{n-2}\,\delta\varphi}{J(\varphi, f)}=\text{const. of }\frac{(AD-BC)_0(x, y, z)_{12}^{n-2}d\varphi_0}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where on the right-hand side $(x, y, z)_{12}^{n-2}$ is considered as a function of ρ, σ, τ, and we are to put therein $\tau=0$; it has been seen (No. 26) that the value is $=\dfrac{f_0\Delta^2}{\rho\sigma}$, where f_0 is what f considered as a function of ρ, σ, τ becomes on writing therein $\tau=0$; the right-hand side thus becomes

$$=\text{const. of }\frac{(AD-BC)_0 f_0\Delta^2\delta\varphi_0}{\alpha\beta(\rho\sigma)^{mn}}$$

41. But for $\tau=0$ we have

$$A_0\phi_0+B_0f_0=\alpha\rho^{mn},$$
$$C_0\phi_0+D_0f_0=\beta\sigma^{mn},$$

and hence $$(AD-BC)_0 f_0 = A_0\beta\sigma^{mn} - C_0\alpha\rho^{mn},$$
and the right-hand side thus becomes, say
$$-\Delta^2 \text{ const. of } \left(-\frac{A_0}{\alpha\rho^{mn}} + \frac{C_0}{\beta\sigma^{mn}}\right)\delta\phi_0.$$
But in calculating the constant of $\frac{A_0}{\alpha\rho^{mn}}\,\delta\phi_0$, we may suppose not only $\tau=0$, but also $\sigma=0$: we then have $\phi_0=(x,\,y,\,z)^m$, $=\left(\frac{\rho}{\varDelta}\right)^m(x_1,\,y_1,\,z_1)^m$, $=\left(\frac{\rho}{\varDelta}\right)^m\phi_1$, and hence also $\delta\phi_0=\left(\frac{\rho}{\varDelta}\right)^m\phi_1$.

Similarly in calculating the constant of $\frac{B_0}{\beta\sigma^{mn}}\delta\phi_0$, we may suppose not only $\tau=0$, but also $\rho=0$, we then have $\phi_0=(x,\,y,\,z)^m$, $=\left(\frac{\sigma}{\varDelta}\right)^m(x_2,\,y_2,\,z_2)^m$, $=\left(\frac{\sigma}{\varDelta}\right)^m\phi_2$, and hence $\delta\phi_0=\left(\frac{\sigma}{\varDelta}\right)^m\delta\phi_2$.

Moreover, in the equations
$$A_0\phi_0 + B_0 f_0 = \alpha\rho^{mn},$$
$$C_0\phi_0 + D_0 f_0 = \beta\sigma^{mn},$$
writing in the first equation $\sigma=0$, we find $A_0\left(\frac{\rho}{\varDelta}\right)^m\phi_1=\alpha\rho^{mn}$, that is $\frac{A_0}{\alpha\rho^{mn}} =\left(\frac{\varDelta}{\rho}\right)^m\frac{1}{\varphi_1}$; and similarly writing in the second equation $\rho=0$, we find $C_0\left(\frac{\sigma}{\varDelta}\right)^m\phi_2=\beta\sigma^{mn}$, that is $\frac{C_0}{\beta\sigma^{mn}}=\left(\frac{\varDelta}{\sigma}\right)^m\frac{1}{\varphi_2}$: and the expression thus becomes
$$= -\Delta^2\left(-\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}\right),$$
giving the equation which was to be proved. This completes the proof of the affected theorem
$$\Sigma\,\frac{(x,\,y,\,z)_{12}^{n-2}\,d\omega}{012} = -\frac{\delta\varphi_1}{\varphi_1}+\frac{\delta\varphi_2}{\varphi_2}.$$

CHAPTER III. THE MAJOR FUNCTION $(x,\,y,\,z)_{12}^{n-2}$.

Analytical Expression of the Function. Art. Nos. 42 to 49.

42. The function has been defined by the conditions that the curve $(x,\,y,\,z)_{12}^{n-2}=0$, shall pass through the dps, and also through the $n-2$ residues of the parametric points $1,\,2$: and moreover, that on writing therein $(x_1,\,y_1,\,z_1)$ for $(x,\,y,\,z)$, the function shall become $=n.1^{n-1}2$. Obviously the function is not completely determined: calling it Ω (or when required Ω_{12}), then if Ω' be any particular form of it, the general form is $\Omega=\Omega'+(x,\,y,\,z)^{n-3}.012$, where

$(x, y, z)^{n-3}$ is the general minor function (viz. $(x, y, z)^{n-3}=0$ is a curve passing through the dps): the major function thus contains $\frac{1}{2}(n-1)(n-2)-\delta, =p$, arbitrary constants.

Agreeing with the definition we have the before-mentioned equation

$$\Omega=\frac{1}{\Delta^{n-2}}(n.1^{n-1}2, \ldots \dagger\between\rho, \sigma)^{n-2}+\&\text{c. } \tau,$$

viz. from this expression for Ω it appears that the curve $\Omega=0$ meets the line through 1, 2 in the $n-2$ residues of these points, and moreover, for $(x, y, z)=(x_1, y_1, z_1)$ and therefore $(\rho, \sigma, \tau)=(\Delta, 0, 0)$, the value of Ω is $=n.1^{n-1}2$.

43. We can without difficulty write down an equation determining Ω' as a function $(x, y, z)^{n-2}$, which on putting therein $\tau=0$, becomes equal to the foregoing expression $\frac{1}{\Delta^{n-2}}(n.1^{n-1}2, \ldots \dagger\between\rho, \sigma)^{n-2}$, and which is moreover such that the curve $\Omega'=0$ passes through the dps; which being so, we have as before, $\Omega=\Omega'+(x, y, z)^{n-3}.012$, for the general value of Ω.

To fix the ideas, consider the particular case $n=4$, the fixed curve a quartic: Ω', on putting therein $\tau=0$, should become

$$=\frac{1}{\Delta^2}(4.1^32,\ 6.1^22^2,\ 4.12^3\dagger\between\rho, \sigma)^2;$$

and it is to be shown that this will be the case if we determine Ω' a quadric function of (x, y, z) by the equation

$$\begin{vmatrix} (x, y, z)^2 & , \Omega' \\ 1(x_1, y_1, z_1)^2 & , 4.1^32 \\ 2(x_1, y_1, z_1\between x_2, y_2, z_2), & 6.1^22^2 \\ 1(x_2, y_2, z_2)^2 & , 4.12^3 \\ a, b, c, f, g, h & , 0 \\ & \\ & \end{vmatrix}=0,$$

where the left-hand side is a determinant of seven lines and columns, the top line being $x^2, y^2, z^2, 2yz, 2zx, 2xy, \Omega'$ and similarly for the second line; the third line is $2x_1x_2,\ 2y_1y_2,\ 2z_1z_2,\ 2(y_1z_2+y_2z_2),\ 2(z_2x_1+z_1x_2),\ 2(x_1y_2+x_2y_1),\ 6.1^22^2$, and in each of the last three lines we have six arbitrary constants followed by a 0. The equation is of the form $\square+M\Omega'=0$, where $\square$ is a quadric function $(x, y, z)^2$, and M is a constant factor.

44. If the quartic curve has a dp, suppose at the point α, coordinates $(x_\alpha, y_\alpha, z_\alpha)$, then in order that the curve $\Omega'=0$ may pass through the dp, we must for one of the last three lines substitute $(x_\alpha, y_\alpha, z_\alpha)^2$, 0; and so for any

other dp or dps of the quartic curve. And the conditions as to the dp or dps (if any) being satisfied in this manner, we may if we please, taking $(x_\beta, y_\beta, z_\beta)$ as the coordinates of an arbitrary point β (not of necessity on the fixed curve), write any line not already so expressed, of the last three lines, in the form $(x_\beta, y_\beta, z_\beta)^2$, 0; the effect being to make the curve $\Omega' = 0$ pass through the arbitrary point β.

45. To show that the equation on putting therein $\tau = 0$ does in fact give the required value, $\Omega' = \frac{1}{\Delta^2}(4.1^3 2,\ 6.1^2 2^2,\ 4.12^3 \dagger\!\!\between \rho, \sigma)^2$, $= \Phi$ suppose, it is to be observed that effecting a linear substitution upon the first six columns, the equation may be written

$$\left|\begin{array}{ll} (\rho, \sigma, \tau)^2 & , \Omega' \\ 1\,(\rho_1, \sigma_1, \tau_1)^2 & , 4.1^3 2 \\ 2\,(\rho_1, \sigma_1, \tau_1 \between \rho_2, \sigma_2, \tau_2), & 6.1^2 2^2 \\ 1\,(\rho_2, \sigma_2, \tau_2)^2 & , 4.12^3 \\ a', b', c', f', g', h' \,, & 0 \end{array}\right| = 0,$$

where $(\rho_1, \sigma_1, \tau_1)$, $(\rho_2, \sigma_2, \tau_2)$ are what (ρ, σ, τ) become on writing therein for (x, y, z) the values (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; viz. we have $(\rho_1, \sigma_1, \tau_1) = (\Delta, 0, 0)$; $(\rho_2, \sigma_2, \tau_2) = (0, \Delta, 0)$; the equation thus is

$$\left|\begin{array}{ccccccc} \rho^2, & \sigma^2, & \tau^2, & 2\sigma\tau, & 2\tau\rho, & 2\rho\sigma, & \Omega' \\ \Delta^2, & 0, & 0, & 0, & 0, & 0, & 4.1^3 2 \\ 0, & 0, & 0, & 0, & 0, & 2\Delta^2, & 6.1^2 2^2 \\ 0, & \Delta^2, & 0, & 0, & 0, & 0, & 4.12^3 \\ a', & b', & c', & f', & g', & h', & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \end{array}\right| = 0,$$

and then by another linear substitution upon the columns, the last column can be changed into $\Omega' - \Phi$, 0, 0, 0, 0, 0, 0; whence writing $\tau = 0$, the equation becomes

$$\left|\begin{array}{ccccccc} \rho^2, & \sigma^2, & 0, & 0, & 0, & 2\rho\sigma, & \Omega' - \Phi \\ \Delta^2, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 2\Delta^2, & 0 \\ 0, & \Delta^2, & 0, & 0, & 0, & 0, & 0 \\ a', & b', & c', & f', & g', & h', & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \end{array}\right| = 0,$$

or, omitting a constant factor, it is

$$\begin{vmatrix} \rho^2, & \sigma^2, & 2\rho\sigma, & \Omega'-\Phi \\ \Delta^2, & 0, & 0, & 0 \\ 0, & 0, & 2\Delta^2, & 0 \\ 0, & \Delta^2, & 0, & 0 \end{vmatrix} = 0,$$

that is $\Omega' - \Phi = 0$, or $\Omega' = \Phi$, $= \frac{1}{\Delta^2}(4.1^32\,,\ 6.1^22^2,\ 4.12^3 \dagger \between \rho,\ \sigma)^2$, the required value.

46. Considering the equation for Ω' as expressed in the before-mentioned form $\square + M\Omega' = 0$, the value of the constant factor M is

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ a,\ b,\ c,\ f,\ g,\ h, \\ \cdot \\ \cdot \end{vmatrix};$$

or if instead of each line such as a, b, c, f, g, h, we have a line $(x_\alpha,\ y_\alpha,\ z_\alpha)^2$ then we have

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ (x_\alpha,\ y_\alpha,\ z_\alpha)^2 \\ (x_\beta,\ y_\beta,\ z_\beta)^2 \\ (x_\gamma,\ y_\gamma,\ z_\gamma)^2 \end{vmatrix},$$

a value which is

$$= \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\alpha, & y_\alpha, & z_\alpha \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\beta, & y_\beta, & z_\beta \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix} \begin{vmatrix} x_\alpha, & y_\alpha, & z_\alpha \\ x_\beta, & y_\beta, & z_\beta \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix},$$

or say this is $= 12\alpha.12\beta.12\gamma.\alpha\beta\gamma$.

47. It is obvious that the foregoing process is applicable to the general case of the fixed curve of the order n with δ dps, and gives always Ω', by an equation of the foregoing form $\square + M\Omega' = 0$, where $\square$ is a function $(x,\ y,\ z)^{n-2}$ of the coordinates, and M is a constant factor. Supposing that in the deter-

minant for Ω', each of the lower lines is written in the form $(x_\alpha, y_\alpha, z_\alpha)^{n-2}$, 0, the number of the points α is $=\frac{1}{2}(n-1)(n-2)$, viz. these are the δ dps, and $\frac{1}{2}(n-1)(n-2)-\delta$, $=p$, other points α. The general expression of M is $M=12\alpha.12\beta\ldots(\alpha^{n-3}\beta^{n-3}\ldots)$, viz. equating to zero a factor such as 12α, this expresses that the point α is on the line 12; but equating to zero the last factor $(\alpha^{n-3}\beta^{n-3}\ldots)$, this expresses that the several points α, viz. the dps and the p other points α, are on a curve of the order $n-3$.

48. Preceding the case $n=4$, above considered, we have, of course, the case $n=3$, $\delta=0$, the fixed curve a cubic; the equation for Ω' is here

$$\begin{vmatrix} x, & y, & z, & \Omega' \\ x_1, & y_1, & z_1, & 3.1^22 \\ x_2, & y_2, & z_2, & 3.12^2 \\ x_\alpha, & y_\alpha, & z_\alpha, & \end{vmatrix}=0,$$

giving

$$\tfrac{1}{3}\Omega'=\frac{1^22.02\alpha+12^2.0\alpha1}{12\alpha},$$

or if we write herein 3 for α, this is

$$\tfrac{1}{3}\Omega'=\frac{1^22.023+12^2.031}{123},$$

and we have hence the general form

$$\tfrac{1}{3}\Omega=\frac{1^22.023+12^2.031}{123}+K.012,$$

where K is an arbitrary constant.

49. There is, however, a more simple particular solution $\frac{1}{3}\Omega'=$ polar function 012 $(f=x^3+y^3+z^3$, then $012=xx_1x_2+yy_1y_2+zz_1z_2)$, which to avoid a confusion of notation, we may write $=\widetilde{012}$. We at once verify this, for expressing the coordinates (x, y, z) in terms of (ρ, σ, τ) we have $\frac{1}{3}\Omega'=\widetilde{012}$, $=\frac{1}{\Delta}(1^22.\rho+12^2.\sigma+\widetilde{123}.\tau)$, which, for $\tau=0$ becomes $=\frac{1}{\Delta}\{1^22.\rho+12^2.\sigma\}$.

We must, of course, have an identity of the form

$$\widetilde{012}=\frac{1^22.023+12^2.031}{123}+K.012,$$

and to find K, writing here $0=3$, we have $K=\frac{\widetilde{123}}{123}$, or we have the identity

$$123\,\widetilde{012}-\widetilde{123}\,012=1^22.023+12^2.031.$$

Single Letter Notation for the Polar Functions of the Cubic.
Art. Nos. 50 nd 51.

50. The notation of single letters for the polar functions is not much required in the case of the cubic, but, in the next following case of the quartic it can hardly be dispensed with, and I therefore establish it in the case of the cubic: viz. I write

$$2^2 3,\ 3^2 1,\ 1^2 2 = f,\ g,\ h,\ 23^2,\ 31^2,\ 12^2 = i,\ j,\ k;\ \widetilde{123} = l,$$

or what is the same thing, the expression for the cubic function f, in terms of $\rho,\ \sigma,\ \tau$ is

$$\Delta^3.f = 3h\rho^2\sigma + 3j\rho^2\tau + 3k\rho\sigma^2 + 6l\rho\sigma\tau + 3g\rho\tau^2 + 3f\sigma^2\tau + 3i\sigma\tau^2;$$

an equation, which writing 0^3 instead of f, may also be written

$$\Delta^3.0^3 = (3h,\ 3j,\ 3k,\ 6l,\ 3g,\ 3f,\ 3i \between \rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^2\tau,\ \sigma\tau^2),$$

and I join to it the series of equations

$$\begin{aligned}
\Delta^2.0^2 1 &= (0,\ 2h,\ 2j,\ k,\ 2l,\ g \between \rho^2,\ \rho\sigma,\ \rho\tau,\ \sigma^2,\ \sigma\tau,\ \tau^2),\\
\text{"}\ \ 0^2 2 &= (h,\ 2k,\ 2l,\ 0,\ 2f,\ i \between \quad \text{"} \quad),\\
\text{"}\ \ 0^2 3 &= (j,\ 2l,\ 2g,\ f,\ 2i,\ 0 \between \quad \text{"} \quad),\\
\Delta.01^2 &= (0,\ h,\ j \between \rho,\ \sigma,\ \tau),\\
\Delta\ \widetilde{012} &= (h,\ k,\ l \between \quad \text{"} \quad),\\
\text{"}\ \ \widetilde{013} &= (j,\ l,\ g \between \quad \text{"} \quad),\\
\text{"}\ \ 02^2 &= (k,\ 0,\ f \between \quad \text{"} \quad),\\
\text{"}\ \ \widetilde{023} &= (l,\ f,\ i \between \quad \text{"} \quad),\\
\text{"}\ \ 03^2 &= (g,\ i,\ 0 \between \quad \text{"} \quad).
\end{aligned}$$

51. In particular we have $\Delta.\widetilde{012} = h\rho + k\sigma + l\tau$, and the above-mentioned identity $123\ \widetilde{012} - \widetilde{123}\ 012 = 1^2 2.023 + 12^2.031$ is simply $h\rho + k\sigma + l\tau - l\tau = h\rho + k\sigma$.

Single Letter Notation for the Polar Functions of the Quartic. Art. No. 52.

52. I write here

$$\begin{aligned}
&2^3 3,\ 3^3 1,\ 1^3 2 = f,\ g,\ h;\quad 23^3,\ 31^3,\ 12^3 = i,\ j,\ k;\\
&1^2 23,\ 12^2 3,\ 123^2 = l,\ m,\ n;\quad 2^2 3^2,\ 3^2 1^2,\ 1^2 2^2 = p,\ q,\ r;
\end{aligned}$$

so that the expression for the quartic function f in terms of $\rho,\ \sigma,\ \tau$ is

$$\begin{aligned}
\Delta^4.f = {} & 4h\rho^3\sigma + 4j\rho^3\tau + 6p\rho^2\sigma^2 + 12l\rho^2\sigma\tau + 6q\rho^2\tau^2\\
& + 4k\rho\sigma^3 + 12m\rho\sigma^2\tau + 12n\rho\sigma\tau^2 + 4g\rho\tau^3 + 4f\sigma^3\tau + 6r\sigma^2\tau^2 + 4i\sigma\tau^3,
\end{aligned}$$

which, putting 0^4 for f, may also be written

$$\begin{aligned}
\Delta^4.0^4 = {} & (4h,\ 4j;\ 6p,\ 12l,\ 6q:\ 4k,\ 12m,\ 12n,\ 4g:\ 4f,\ 6r,\ 4i \dagger)\\
& (\rho^3\sigma,\ \rho^3\tau;\ \rho^2\sigma^2,\ \rho^2\sigma\tau,\ \rho^2\tau^2,\ \rho\sigma^3,\ \rho\sigma^2\tau,\ \rho\sigma\tau^2,\ \rho\tau^3,\ \sigma^3\tau,\ \sigma^2\tau^2,\ \sigma\tau^3),
\end{aligned}$$

and I join to it the series of equations

$$
\begin{array}{lll}
\Delta^3.0^31 & = (0;\ 3h,\ 3j;\ 3r,\ 6l,\ 3q;\ k,\ 3m,\ 3n,\ g)\!(\rho^3,\ \rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^3,\ \sigma^2\tau,\ \sigma\tau^2,\ \tau^3), \\
\text{"}\quad 0^32 & = (h;\ 3r,\ 3l;\ 3k,\ 6m,\ 3n;\ 0,\ 3f,\ 3p,\ i)\!(\quad \text{"} \quad), \\
\text{"}\quad 0^33 & = (j;\ 3l,\ 3q;\ 3n,\ 6n,\ 3g;\ f,\ 3p,\ 3i,\ 0)\!(\quad \text{"} \quad), \\
\Delta^2.0^21^2 & = (0;\ 2h,\ 2j;\ r,\ 2l,\ q)\!(\rho^2;\ \rho\sigma,\ \rho\tau;\ \sigma^2,\ \sigma\tau,\ \tau^2), \\
\text{"}\quad 0^212 & = (h;\ 2r,\ 2l;\ k,\ 2m,\ n)\!(\quad \text{"} \quad), \\
\text{"}\quad 0^213 & = (j;\ 2l,\ 2q;\ m,\ 2n,\ g)\!(\quad \text{"} \quad), \\
\text{"}\quad 0^22^2 & = (r;\ 2k,\ 2m;\ 0,\ 2f,\ p)\!(\quad \text{"} \quad), \\
\text{"}\quad 0^223 & = (l;\ 2m,\ 2n;\ f,\ 2p,\ i)\!(\quad \text{"} \quad), \\
\text{"}\quad 0^23^2 & = (q;\ 2n,\ 2g;\ p,\ 2i,\ 0)\!(\quad \text{"} \quad), \\
\Delta.01^3 & = (0,\ h,\ j)\!(\rho,\ \sigma,\ \tau), \\
\text{"}\quad 01^22 & = (h,\ r,\ l)\!(\quad \text{"} \quad), \\
\text{"}\quad 01^23 & = (j,\ l,\ q)\!(\quad \text{"} \quad), \\
\text{"}\quad 012^2 & = (r,\ k,\ m,)\!(\quad \text{"} \quad), \\
\text{"}\quad 0123 & = (l,\ m,\ n)\!(\quad \text{"} \quad), \\
\text{"}\quad 013^2 & = (q,\ n,\ g)\!(\quad \text{"} \quad), \\
\text{"}\quad 02^3 & = (k,\ 0,\ f)\!(\quad \text{"} \quad), \\
\text{"}\quad 02^23 & = (m,\ f,\ p)\!(\quad \text{"} \quad), \\
\text{"}\quad 023^2 & = (n,\ p,\ i)\!(\quad \text{"} \quad), \\
\text{"}\quad 03^3 & = (g,\ i,\ 0)\!(\quad \text{"} \quad),
\end{array}
$$

which will be convenient in the sequel.

Major Function—The Fixed Curve a Cubic. Art. No. 53.

53. It has been already seen that a simple particular form is $\frac{1}{3}\,\Omega' = \widetilde{012}$: and that the general form is $\Omega = \Omega' + K.012$.

Major Function—The Fixed Curve a Quartic. Art. No. 54.

54. It is to be shown that a particular form is

$$\tfrac{1}{2}\,\Omega' = \frac{-01^3.02^3 + 01^22.012^2 + 0^212.1^22^2}{1^2\,2^2}.$$

In fact by the foregoing values of $\Delta.01^3$, etc., the numerator of this expression, multiplied by Δ^2 is =

$$
\begin{aligned}
&-(h\sigma + j\tau)(k\rho + f\tau) \\
&+ (h\rho + r\sigma + l\tau)(r\rho + k\sigma + m\tau) \\
&+ r(h\rho^2 + 2r\rho\sigma + 2l\rho\tau + k\sigma^2 + 2m\sigma\tau + n\tau^2)
\end{aligned}
$$

which is

$$= 2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau$$
$$+ 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2$$

and this for $\tau = 0$, becomes

$$= r(2h\rho^2 + 3r\rho\sigma + 2k\sigma^2).$$

Hence for $\tau = 0$, we have

$$\Omega' = \frac{1}{\Delta^2}(4h\rho^2 + 6r\rho\sigma + 4k\sigma^2), \text{ that is}$$
$$= \frac{1}{\Delta^2}\{4.1^3 2.\rho^2 + 6.1^2 2^2.\rho\sigma + 4.12^3.\sigma^2\}$$

and Ω' is thus a form of the major function $(x, y, z)^2_{12}$. Of course the general form is $\Omega = \Omega' + (x, y, z)^1.012^1$.

Syzygy of the Major Function. Art. No. 55.

55. Writing now $(x, y, z)^{n-2}_{12} = \Omega_{12}$; and taking on the fixed curve a new point 3, consider the like functions Ω_{23} and Ω_{31}: it is to be shown that we have identically

$$\Omega_{23}.031.012 + \Omega_{31}.012.023 + \Omega_{12}.023.031 - (123)^2 f = 023.031.012\,(x, y, z)^{n-3},$$

where $(x, y, z)^{n-3}$ is a *properly determined* minor function. Or considering herein 0 as a point on the fixed curve and writing therefore $f = 0$, the equation is

$$\frac{\Omega_{23}}{023} + \frac{\Omega_{31}}{031} + \frac{\Omega_{12}}{012} = (x, y, z)^{n-3}. \quad \text{(See footnote*.)}$$

56. Write for a moment $X = \Omega_{23}.031.012 + \Omega_{31}.012.023 + \Omega_{12}.023.031$, then k being an arbitrary coefficient, we have $X - kf = 0$, a curve of the order n, passing through the points 1, 2, 3, and also through the residues of 2, 3, the residues of 3, 1, and the residues of 1, 2; in fact at the point 1 we have $012 = 0$, $031 = 0$, and therefore $X = 0$; also $f = 0$; and therefore 1 is a point on the curve. Again at any residue of 2, 3 we have $\Omega_{23} = 0$, $023 = 0$, and therefore $X = 0$; also $f = 0$; and hence the residue of 2, 3 is a point on the curve.

It is next to be shown that k can be so determined that the curve $X - kf = 0$ shall have a dp at each of the points 1, 2, 3. Supposing this to be so, we have the line 23 meeting the curve $X - kf = 0$ in the points 2 and 3, each counting twice, and in the $n - 2$ residues of 2, 3, that is in $n + 2$ points; hence the curve $X - kf = 0$ must contain as part of itself the line 23, and

* This is the differential theorem corresponding to C. and G.'s integral theorem, p. 26, viz. this is $S_{\xi\eta} + S_{\eta\zeta} + S_{\zeta\xi} = I$, a sum of three integrals of the third kind = an integral of the first kind.

similarly it must contain as part of itself each of the other lines 31 and 12, viz. we shall then have $X - kf = 023.031.012.(x, y, z)^{n-3}$; and from this equation observing that the curves $\Omega_{23} = 0$, $\Omega_{31} = 0$, $\Omega_{12} = 0$ each pass through the dps, it follows that the curve $(x, y, z)^{n-3} = 0$ also passes through the dps; hence, k being found to be $= (123)^2$, the theorem will be proved.

57. Taking an arbitrary point α coordinates $(x_\alpha, y_\alpha, z_\alpha)$, and writing $D = x_\alpha \frac{d}{dx} + y_\alpha \frac{d}{dy} + z_\alpha \frac{d}{dz}$ we have to find k, so that the curve $D(X - kf) = 0$ shall pass through the point 1. Observing that $D023 = \alpha 23$, etc., we have

$$\begin{aligned} D(X - kf) = {} & D\Omega_{23}.031.012 + \Omega_{23}(031.\alpha 12 + \alpha 31.012) \\ & + \alpha 23\,(\Omega_{31}.012 + \Omega_{12}.031) \\ & + 023\{\Omega_{31}.\alpha 12 + \Omega_{12}.\alpha 31 + D\Omega_{31}.012 + D\Omega_{12}.031\} \\ & - kDf, \end{aligned}$$

and, to make the curve pass through 1, writing herein $0 = 1$, we have

$$0 = 123\,(\Omega^1_{31}.\alpha 12 + \Omega^1_{12}.\alpha 31) - k(Df)^1,$$

where the superfix (1) denotes that we are in Ω_{31}, Ω_{12} and Df respectively to write $0 = 1$. We have $\Omega^1_{31} = n.1^{n-1}3$, $\Omega^1_{12} = n.1^{n-1}2$, $(Df)^1 = n.1^{n-1}\alpha$, and the equation thus is

$$n.123\,(1^{n-1}3.\alpha 12 + 1^{n\ \ 1}2.\alpha 31) - kn.1^{n-1}\alpha = 0.$$

But we have identically $1^{n-1}1.\alpha 23 + 1^{n-1}2.\alpha 31 + 1^{n-1}3.\alpha 12 = 1^{n-1}\alpha.123$, where $1^{n-1}1, = 1^n$ is in fact $= 0$; the factor $1^{n-1}\alpha$ thus divides out, and the equation becomes $k = (123)^2$; viz. k having this value, the curve $X - kf = 0$ will have a dp at 1; and clearly by symmetry, it will also have a dp at 2, and at 3; the theorem is thus proved.

The Syzygy, Fixed Curve a Cubic. Art. No. 58.

58. The syzygy may be verified independently in the case where the fixed curve is a cubic. Observe that the syzygy, if satisfied for any particular form of Ω will be generally satisfied; we may therefore take $\frac{1}{3}\Omega_{12} = \widetilde{012}$.

Writing then
$$\frac{\frac{1}{3}\Omega_{12}}{012} = \frac{\widetilde{012}}{012}, = \{012\} \text{ suppose},$$

and taking 0 to be a point on the cubic curve, we ought to have $\{023\} + \{031\} + \{012\} =$ a constant; the value of this constant comes out to be $= \{123\}$, and the syzygy in its complete form thus is

$$\{023\} + \{031\} + \{012\} = \{123\}.$$

We have

$$\Delta\,\widetilde{023},\ \Delta\,\widetilde{031},\ \Delta\,\widetilde{012} = l\rho + f\sigma + i\tau,\ j\rho + l\sigma + g\tau,\ h\rho + k\sigma + l\tau,$$

and the equation thus is

$$\frac{l\rho + f\sigma + i\tau}{\rho} + \frac{j\rho + l\sigma + h\tau}{\sigma} + \frac{h\rho + k\sigma + l\tau}{\tau} - l = 0\,;$$

this, multiplied by $\rho\sigma\tau$ becomes

$$h\rho^2\sigma + j\rho^2\tau + k\rho\sigma^2 + 2l\rho\sigma\tau + g\rho\tau^2 + f\sigma^2\tau + i\sigma\tau^2 = 0,$$

which is in fact $\frac{1}{3}f = 0$, the equation of the cubic curve.

Observe that the new symbol $\{012\}$ is in virtue of its determinant denominator, an alternate function, $\{012\} = -\{102\}$, $\{012\} = \{120\} = \{201\}$. The syzygy is a relation between any four points 1, 2, 3, 0 of the curve, and it may be also expressed in the form

$$\{123\} - \{230\} + \{301\} - \{012\} = 0.$$

The Syzygy, Fixed Curve a Quartic. Art. No. 59.

59. Taking Ω_{12} as before, we have

$$\frac{\frac{1}{2}\Omega_{12}}{012} = \frac{-01^3.02^3 + 01^2 2.012^2 + 0^2 12.1^2 2^2}{012.1^2 2^2} = \{0^2 12\} \text{ suppose}:$$

and then taking 0 to be a point on the quartic curve, we ought to have

$$\{0^2 23\} + \{0^2 31\} + \{0^2 12\} = (x,\,y,\,z)^1 \text{ a linear function of } (x,\,y,\,z),$$

or what is the same thing, considering the left-hand side as expressed in terms of ρ,σ,τ, the sum should be

$$= (\rho,\,\sigma,\,\tau)^1, \text{ a linear function of } (\rho,\,\sigma,\,\tau).$$

By a preceding formula we have

$$\begin{aligned}\{0^2 12\} = \frac{1}{\Delta^2 r\tau}\{&2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau \\ &+ 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2\},\end{aligned}$$

which is

$$\begin{aligned}= \frac{1}{\Delta^2}\left\{\left(3l + \frac{hm - jk}{r}\right)\rho + \left(3m + \frac{-fh + kl}{r}\right)\sigma + \left(n + \frac{-fj + lm}{r}\right)\tau\right\} \\ + \frac{1}{\Delta^2}\frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}.\end{aligned}$$

And hence forming the sum $\{0^2 23\} + \{0^2 31\} + \{0^2 12\}$, we have first a fractional part which is found to be integral, viz. this is

$$\frac{1}{\Delta^2}\left\{\frac{2f\sigma^2 + 3p\sigma\tau + 2i\tau^2}{\rho} + \frac{2g\tau^2 + 3q\tau\rho + 2j\rho^2}{\sigma} + \frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}\right\},$$

$$= \frac{1}{\Delta^2\rho\sigma\tau}\{2h\rho^3\sigma + 2j\rho^3\tau + 3r\rho^2\sigma^2 + 3q\rho^2\tau^2 + 2k\rho\sigma^3 + 2g\rho\tau^3 + 2f\sigma^3\tau + 3p\sigma^2\tau^2 + 2i\sigma\tau^3\},$$

$$= \frac{1}{\Delta^2\rho\sigma\tau}\{\tfrac{1}{2}\Delta^4 f - 6l\rho^2\sigma\tau - 6m\rho\sigma^2\tau - 6n\rho\sigma\tau^2\},$$

or since $f = 0$, this is $= \frac{1}{\Delta^2}(-6l\rho - 6m\sigma - 6n\tau)$,

and then integral terms which are at once deduced from the above integral terms of $0^2 12$; and collecting the several terms we find

$$\{0^2 23\} + \{0^2 31\} + \{0^2 12\} =$$

$$\frac{1}{\Delta^2}\left\{\rho\left(l + \frac{mn - gk}{p} + \frac{jn - gh}{q} + \frac{hm - jk}{r}\right) + \sigma\left(m + \frac{fn - ik}{p} + \frac{ln - hi}{q} + \frac{kl - fh}{r}\right)\right.$$
$$\left. + \tau\left(n + \frac{lm - fg}{p} + \frac{gl - ij}{q} + \frac{lm - fj}{r}\right)\right\}$$

which is the required result.

Preparation for the Conversion—The Symbol ∂. Art. Nos. 60 to 63.

60. I use ∂ as the symbol of a quasi-differentiation, viz., U being any function of (x, y, z), ∂U denotes $\frac{1}{d\omega}$ into the differential $\frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dz}dz$; in such a differential the increments dx, dy, dz do not in general present themselves in the combinations $ydz - zdy$, $zdx - xdz$, $xdy - ydx$; but they will do so if U is a function of the degree zero in the coordinates x, y, z (that is, if U be the quotient of two homogeneous functions of the same degree); and this being so, we can by the equations

$$\frac{ydz - zdy}{\frac{df}{dx}} = \frac{zdx - xdz}{\frac{df}{dy}} = \frac{xdy - ydx}{\frac{df}{dz}}, = d\omega$$

get rid of the increments, and ∂U will denote a function of (x, y, z) derived in a definite manner from the function U; the symbol ∂ will be used only in the case in question of a function of the degree zero. Of course ∂_1 will denote the like operation in regard to (x_1, y_1, z_1); and so ∂_2, etc.; and we may for greater clearness write ∂_0 in place of ∂.

61. Consider then $\partial\frac{P}{Q}$, where P, Q are functions $(x, y, z)^m$ of the same degree, we have

$$\partial\frac{P}{Q} = \frac{1}{Q^2 d\omega}(QdP - PdQ),$$

and then

$$dP = \frac{dP}{dx}dx + \frac{dP}{dy}dy + \frac{dP}{dz}dz, \quad \frac{1}{m}P = \frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z,$$

with the like formulæ for Q. Substituting, we find

$$\partial\frac{P}{Q}=\frac{1}{mQ^2d\omega}\left\{\frac{d(Q,P)}{d(y,z)}(ydz-zdy)+\frac{d(Q,P)}{d(z,x)}(zdx-xdz)+\frac{d(Q,P)}{d(x,y)}(xdy-ydx)\right\},$$

that is

$$\partial\frac{P}{Q}=\frac{1}{mQ^2}\left\{\frac{df}{dx}\frac{d(Q,P)}{d(y,z)}+\&\text{c.}\right\}=\frac{1}{mQ^2}\frac{d(f,Q,P)}{d(x,y,z)},=\frac{1}{mQ^2}J(f,\ Q,\ P),$$

or say

$$\partial\frac{P}{Q}=-\frac{1}{mQ^2}J(P,\ Q,f).$$

62. As an example consider

$$\partial\{012\},=-\frac{1}{(012)^2}J(\widetilde{012},\ 012,f).$$

The determinant is

$$\left|\begin{array}{lll}\dfrac{d}{dx}\widetilde{012}, & y_1z_2-y_2z_1, & \dfrac{df}{dx}\\ \dfrac{d}{dy}\widetilde{012}, & z_1x_2-z_2x_1, & \dfrac{df}{dy}\\ \dfrac{d}{dz}\widetilde{012}, & x_1y_2-x_2y_1, & \dfrac{df}{dz}\end{array}\right|,$$

and the coefficient herein of $\dfrac{d}{dx}\widetilde{012}$ is $(z_1x_2-z_2x_1)\dfrac{df}{dz}-(x_1y_2-x_2y_1)\dfrac{df}{dy}$, which is

$$=x_2\left(x_1\frac{df}{dx}+y_1\frac{df}{dy}+z_1\frac{df}{dz}\right)-x_1\left(x_2\frac{df}{dx}+y_2\frac{df}{dy}+z_2\frac{df}{dz}\right),=3\,(0^21.x_2-0^22.x_1);$$

and so for the other terms.

The determinant is thus

$$=3\left[0^21\left(x_2\frac{d}{dx}+y_2\frac{d}{dy}+z_2\frac{d}{dz}\right)-0^22\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}+z_1\frac{d}{dz}\right)\right]\widetilde{012}$$

say this is

$$=3\,[0^21\,.\,D_{\substack{2\\0}}-0^22\,D_{\substack{1\\0}}]\,\widetilde{012}.$$

But we have $D_{\substack{2\\0}}\,\widetilde{012}=12^2$, $D_{\substack{1\\0}}\,\widetilde{012}=1^22$, and the determinant is then $=3\,(0^21\,.\,12^2-0^22\,.\,1^22)$; whence finally writing ∂_0 instead of ∂

$$\partial_0\{012\}=-3\,.\,\frac{0^21\,.\,12^2-0^22\,.\,1^22}{(012)^2}.$$

63. By cyclical interchange of the $0,1,2$, we have

$$\partial_1\{012\}=-3\,.\,\frac{1^22\,.\,0^22-01^2\,.\,02^2}{(012)^2},$$

$$\partial_2\{012\}=-3\,.\,\frac{02^2\,.\,01^2-12^2\,.\,0^21}{(012)^2};$$

and thence adding, we find

$$(\partial_0 + \partial_1 + \partial_2)\{012\} = 0,$$

an important property, which joined to the equation before obtained,

$$\{023\} + \{031\} + \{012\} = \{123\},$$

completes the theory of the function $\{012\}$.

Conversion of the Major Function (Interchange of Limits and Parametric Points). Art. No. 64.

64. Write in general

$$\frac{(x, y, z)_{12}^{n-2}}{012} = Q_{0,12},$$

$Q_{0,12}$ is an alternate function in regard to the points $1, 2\,(Q_{0,12} = -Q_{0,21})$, and it is in regard to the coordinates of the points $0, 1, 2$, rational, but not integral, of the degrees $n-3, 0, 0$ respectively: it can therefore be operated upon with ∂_1 or ∂_2, but (except in the case $n = 3$) not with ∂_0.

The conversion relates not to the general major function $(x, y, z)_{12}^{n-2}$, but to this function *with the arbitrary constants properly determined*, and consists in a relation between two functions $Q_{4,12}$ and $Q_{1,45}$ (each of them a function of three out of four arbitrary points $1, 2, 4, 5$ on the fixed curve), viz. the conversion is

$$\partial_1 Q_{4,12} = \partial_4 Q_{1,45},$$

an equation which may be written in four different forms, viz. we may in the form written down interchange $1, 2$ and also $4, 5$.*

The determination of the constants is a very peculiar one, inasmuch as it is not algebraical, viz. in the case of the cubic curve, about to be considered, it appears that $Q_{0,12}$ contains the term $\int_2^1 d\omega\, \partial_3\{036\}$, which is a transcendental function of the coordinates of the parametric points 1 and 2.

The Conversion, Fixed Curve a Cubic. Art. No. 65.

65. We may write $Q_{0,12} = \{012\} + K$, where K is a constant, that is, it is independent of the point 0, but depends on the parametric points 1 and 2. I assume K to be properly determined, and give an *à posteriori* verification of the

* The meaning of the property is better seen from the integral form: $Q_{0,12}$ is a function of the points $0, 1, 2$ and $Q_{0,45}$ the like function of the points $0, 4, 5$ such that $\int_5^4 d\omega\, Q_{0,12} = \int_2^1 d\omega\, Q_{0,45}$: which equation operated upon with $\partial_1 \partial_4$ gives the formula of the text. And there is thus the meaning (alluded to in the heading) that there exists for the integral of the third kind a canonical form (C. and G.'s endliche Normalform), such that the integral is not altered by the interchange of the limits and the parametric points. The expression for $Q_{0,12}$ mentioned further on in the text for the case, fixed curve a cubic, shows that in this case the canonical form of the integral of the third kind is $\int^4 d\omega \Big[\{012\} + \Big(\int_2^1 d\omega\, \partial_3\{036\} - \{123\}\Big)\Big]$.

equation $\partial_1 Q_{4,12} = \partial_4 Q_{1,45}$. The value is $K = \int_2^1 d\omega\, \partial_3 \{036\} - \{123\}$, where 3, 6 are arbitrary points on the cubic curve, and where in the definite integral, regarded as an integral $\int U du$ with a current variable u, the meaning is that this variable has at the limits the values u_1, u_2 which belong to the points 1 and 2 respectively: a fuller explanation might be proper, but the investigation will presently be given in a form not depending on any integral at all.

Substituting for K its value we have

$$Q_{0,12} = \{012\} + \left[\int_2^1 d\omega \partial_3 \{036\} - \{123\}\right],$$

or as this may also be written

$$= - \{023\} - \{031\} + \int_2^1 d\omega \partial_3 \{036\}.$$

We have thence

$$\partial_1 Q_{0,12} = - \partial_1 \{031\} + \partial_3 \{136\},$$

and consequently

$$\partial_1 Q_{4,12} = - \partial_1 \{431\} + \partial_3 \{136\},$$
$$\partial_4 Q_{1,45} = - \partial_4 \{134\} + \partial_3 \{436\},$$

and hence observing that $\{431\} = - \{134\}$ &c., we have

$$\partial_1 Q_{4,12} - \partial_4 Q_{1,45} = (\partial_1 + \partial_4) \{134\} + \partial_3 \{136\} - \partial_3 \{436\},$$
$$= - \partial_3 \{134\} + \partial_3 \{136\} - \partial_3 \{436\},$$

which observing that we have $\partial_3 \{641\} = 0$ is

$$= \partial_3 (\{136\} - \{364\} + \{641\} - \{413\}), = 0,$$

the required theorem.

To avoid, in the proof, the use of the integral sign, we have only to consider the required function $Q_{0,12}$ as given by the foregoing differential formula

$$\partial_1 Q_{0,12} = - \partial_1 \{031\} + \partial_3 \{136\},$$

for we have then the values of $\partial_1 Q_{4,12}$ and $\partial_4 Q_{1,45}$, and the rest of the proof the same as before.

The Conversion, Fixed Curve a Quartic. Art. Nos. 66 to 73.

66. We have

$$Q_{0,12} = \{0^2 12\} + (x, y, z)^1,$$

where $(x, y, z)^1$ is a linear function of (x, y, z), but depending also on the parametric points 1 and 2, which is to be determined so as to satisfy the conversion equation

$$\partial_1 Q_{4,12} = \partial_4 Q_{1,45}.$$

Observing that we have $\{0^2 23\} + \{0^2 31\} + \{0^2 12\} =$ a linear function of (x, y, z), the linear function $(x, y, z,)^1$ of $Q_{0,12}$ may be taken to be $= \Theta_{0,12} - \{0^2 23\} - \{0^2 31\} - \{0^2 12\}$; that is, we may assume

$$Q_{0,12} = \{0^2 12\} + \Theta_{0,12} - (\{0^2 23\} + \{0^2 31\} + \{0^2 12\}),$$
$$= - \{0^2 23\} - \{0^2 31\} + \Theta_{0,12},$$

where $\Theta_{0,12}$ is a linear function of (x, y, z), but depending also on the points 1 and 2, which has to be determined. We have

$$\partial_1 Q_{0,12} = - \partial_1\{0^2 31\} + \partial_1\Theta_{0,12},$$

and thence

$$\partial_1 Q_{4,12} = - \partial_1\{4^2 31\} + \partial_1\Theta_{4,12},$$
$$\partial_4 Q_{1,45} = - \partial_4\{1^2 34\} + \partial_4\Theta_{1,45},$$

giving an equation for Θ,

$$\partial_1\Theta_{4,12} - \partial_4\Theta_{1,45} = \partial_1\{4^2 31\} - \partial_4\{1^2 34\};$$

4 is here an arbitrary point of the quartic, and we may instead of it write 0, the equation thus becomes

$$\partial_1\Theta_{0,12} - \partial_0\Theta_{1,05} = \partial_1\{0^2 31\} - \partial_0\{1^2 30\}.$$

67. Of the terms on the left-hand side, the first is a linear function of (x, y, z), or say it is an integral function 0^1, and the second is a linear function of (x_1, y_1, z_1), or say it is an integral function 1^1: the given function on the right-hand side must therefore admit of expression in the form $\phi(0^1, 1, 3) - \phi(1^1, 0, 3)$, where $\phi(0^1, 1, 3)$ is a known function, integral and linear as regards the coordinates (x, y, z) of the point 0, but depending also on the points 1, 3; and $\phi(1^1, 0, 3)$ is the like known function, integral and linear as regards the coordinates (x_1, y_1, z_1) of the point 1, but depending also on the points 0, 3. Moreover, since 2 and 5 are arbitrary points entering only on the left-hand side, it is clear that $\partial_1\Theta_{0,12}$ must be independent of 2, and $\partial_0\Theta_{1,05}$ independent of 5 [reverting to the cubic case observe that here $\Theta_{0,12} = \int_2^1 d\omega\partial_3\{036\}$, whence $\partial_1\Theta_{1,12} = \partial_3\{136\}$, and so $\partial_0\Theta_{1,05} = \partial_3\{036\}$, and that the corresponding equation thus is $\partial_3\{136\} - \partial_3\{036\} = \partial_1\{031\} - \partial_0\{130\}$, where the left-hand side is $= \partial_3\{013\}$, and the equation itself $(\partial_0 + \partial_3 + \partial_1)\{031\} = 0$]. We then have

$$\partial_1\Theta_{0,12} - \phi(0^1, 1, 3) = \partial_0\Theta_{1,02} - \phi(1^1, 0, 3),$$

where the one side is derived from the other by the interchange of the 0, 1. The solution therefore is

$$\partial_1\Theta_{0,12} - \phi(0^1, 1, 3) = X(\overline{0, 1}, 3),$$

a function symmetrical in regard to the points 0 and 1, and which, inasmuch as the left-hand is an integral function 0^1, must itself be an integral function $(0^1, 1^1)$, that is, integral and linear as regards the coordinates (x, y, z) and (x_1, y_1, z_1) of the points 0 and 1 respectively. We thus have

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3),$$

and thence

$$\partial_2\Theta_{0,12} = -\phi(0^1, 2, 3) - X(\overline{0, 2}, 3),$$

viz. the second of these expressions is with its sign reversed the same function of 2 that the first is of 1.

68. It follows that taking a new symbol 7 for the variable of the definite integral (in the cubic case $\Theta_{0,12}$ was independent of 0, and there was nothing to prevent the use of 0 for the current point of the definite integral), we may write $\Theta_{0,12} = \int_2^1 d\omega_7 P(7, 0, 3)$, where $\partial_1 P(1, 0, 3) = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3)$, an equation which implies $\partial_2 P(0, 2, 3) = \phi(0^1, 2, 3) + X(\overline{0, 2}, 3)$. But the first of these equations in P is nothing else than the first of the equations in $\Theta_{0,12}$.

70. I have succeeded in finding $\phi(0^1, 1, 3)$, but the calculation is a very tedious one, and I give only the principal steps, omitting all details. We have to bring $\partial_1 0^2 13 - \partial_0 1^2 03$ into the form $\phi(0^1, 1, 3) - \phi(1^1, 0, 3)$. From the value of $\{0^2 13\}, = \dfrac{-01^3.03^3 + 01^2 3.013^2 - 0^2 13.1^2 3^2}{013.1^2 3^2}$, we find by a process such as that of No. 62,

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\sigma^2}\Bigg\{ & -0^2 3^2.01^3 - 0^3 3.j \\ & + \frac{1}{q}\left(\begin{matrix} 2.01^3[03^3.01^2 3 - (013^2)^2] \\ + [2.0^2 13.013^2 + 1.0^2 3^2.01^2 3 - 3.0^2 1^2.0^2 3^2]j \end{matrix}\right) \\ & + \frac{1}{q^2}\left(\begin{matrix} 2.01^3(-01^3.03^3 + 013^2.01^2 3)g \\ -2.013^2(-01^3.03^3 + 013^2.01^2 3)j \end{matrix}\right)\Bigg\}.\end{aligned}$$

Substituting herein the values $01^3 = \frac{1}{\Delta}(h\sigma + j\tau)$ &c. we have $\frac{1}{\sigma^2}$ into a cubic function $(\rho, \sigma, \tau)^3$, and writing down first the integral terms, and then the others, we have

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\Delta^3}\Bigg\{ & \rho\Big[(-6hn + 5jm) + \frac{1}{q}(4ghl - 3gjr - 2hij + j^2 p + 2jnl) \\ & \qquad + \frac{1}{q^2}(-2g^2h^2 + 4ghjn - 2j^2n^2)\Big] \\ & + \sigma\Big[fj - hp + \frac{1}{q}(2hil - 2hn^2 - 3ijr + jlp + 2jmn) \\ & \qquad + \frac{1}{q^2}(2ghln - 2gh^2 i + 2hijn - 2jln^2)\Big]\end{aligned}$$

$$+\tau\Big[3jp+\frac{1}{q}(-2ghn+2gjm-2ijl-2jn^2)$$
$$+\frac{1}{q^2}(2g^2hl-2ghij-2gjln+2ij^2n)\Big]$$

(say this linear function of ρ, σ, τ is $=\square$).

$$+\frac{1}{\Delta^3\sigma^2}\{\rho^3.2j^2+\rho^2\sigma(6jl-3hg)+\rho^2\tau.3jq+\rho\sigma\tau(-2gh+6jn)+\rho\tau^2.2gj+\sigma\tau^2.2ij\}.$$

71. The expression of $\partial_0\{1^203\}$ is deduced from this by the interchange of 0, 1: and I write

$$\partial_1\{0^213\}-\partial_0\{1^203\}=\square-*$$
$$+\frac{1}{\sigma^2\Delta^3\rho^3}[\rho^3\{\rho^3.2j^2+\rho^2\sigma(-3hq+6jl)+\rho^2\tau.3jq$$
$$+\rho\sigma\tau(-2gh+6jn)+\rho\tau^2.2gj+\sigma\tau^2.2ij\}$$
$$-\Delta^3\{\Delta^3.2(0^23^2)^2-\Delta^2\sigma(-3.0^32.0^23^2+6.0^33.0^223)-\Delta^2\tau.3.0^33.0^23^2$$
$$+\Delta\sigma\tau(-2.03^3.0^32+6.0^33.023^2)+\Delta\tau^2.2.03^3.0^33-\sigma\tau^2.2i.0^33\}],$$

where, and in what follows, the $*$ denotes the function immediately to the left of it, interchanging therein the 0, 1. It will be observed that the $\square$, *quâ* linear function of (ρ, σ, τ), that is of (x, y, z), is a term of the required function $\phi(0^1, 1, 3)$: the remaining portion has to be reduced by means of the expressions for $\Delta^2(0^2 3^2)$ etc. in terms of ρ, σ, τ.

72. We obtain $\quad\partial_1\{0^213\}-\partial_0\{1^203\}=\square-*$

$$+\frac{1}{\Delta^3}\{\sigma(2fj-3hp-3kq+18lm-9nr)+\tau(-3gr-3jp+9mq)\}$$
$$+\frac{1}{\Delta^3\rho}\{\sigma^2(12fl-12kn+18m^2-9pr)+\sigma\tau(6fg-6gk-6ir+18mn)+\tau^2.6gm\}$$
$$+\frac{1}{\Delta^3\rho^2}\{\sigma^3(18fm-9kp)+\sigma^2\tau(12fn-8ik+9mp)+\sigma\tau^2(4fg+6im)\}$$
$$+\frac{1}{\Delta^3\rho^3}\{\sigma^4.4f^2+\sigma^3\tau.6fp+\sigma^2\tau^2.4fi\}.$$

The terms of the second line may be transformed as follows:

$$\frac{\sigma}{\Delta^3}(2fj-3hp-3kq+18lm-9nr)$$
$$=\tfrac{1}{2}\frac{\sigma}{\Delta^3}(2fj-3hp-3kq+18lm-9nr)-*$$
$$+\frac{1}{\Delta^3\rho}\{\sigma^2(-12fl+12kn-18m^2+9pr)+\sigma\tau(-\tfrac{3}{2}fq+3gk+\tfrac{9}{2}ir-\tfrac{9}{2}lp-9mn)\}$$
$$+\frac{1}{\Delta^3\rho^2}\{\sigma^3(-30fm+15kp)+\sigma^2\tau(-12fn+12ik-18mp)+\sigma\tau^2.-9np\}$$
$$+\frac{1}{\Delta^3\rho^3}\{\sigma^4.-10f^2+\sigma^3\tau.-15fp+\sigma^2\tau^2.-9p^2+\sigma\tau^3.-3ip\}$$

and

$$\frac{\tau}{\Delta^3}(-3gr-3jp+9mq)$$

$$= \tfrac{1}{2}\frac{\tau}{\Delta^3}(-3gr-3jp+9mq) - *$$
$$+\frac{1}{\Delta^3\rho}\{\sigma\tau(-\tfrac{9}{2}fq+3gk+\tfrac{3}{2}ir+\tfrac{9}{2}lp-9mn)+\tau^2.-6gm\}$$
$$+\frac{1}{\Delta^3\rho^2}\{\sigma^2\tau(-9fn+3ik)+\sigma\tau^2(-6fg-6im)+\tau^3.-3gp\}$$
$$+\frac{1}{\Delta^3\rho^3}\{\sigma^3\tau.-3fp+\sigma^2\tau^2.-6fi+\sigma\tau^3.-3ip\}$$

and substituting these values, the whole third line is destroyed, and we find

$$\partial_1\{0^2 13\}-\partial_0\{1^2 03\}=\square - *$$
$$+\tfrac{1}{2}.\frac{1}{\Delta^3}\{\sigma(2fj-3hq-3kj+18lm-9nr)+\tau(-3gr-3jp+9mq)\}-*$$
$$+\frac{1}{\Delta^3\rho^2}\{\sigma^3(-12fm+6kp)+\sigma^2\tau(-9fn+7ik-9mp)+\sigma\tau^2(-2fg-9np)+\tau^3.-3gp\}$$
$$+\frac{1}{\Delta^3\rho^3}\{\sigma^4.-6f^2+\sigma^3\tau.-12fp+\sigma^2\tau^2(-2fi-9p^2)+\sigma\tau^3.-6ip\}.$$

And ultimately the last two lines of this expression are found to be

$$=\frac{1}{\Delta^3}\{\rho(-2hn+4jm+2l^2-2qr)+\sigma(-2fj+2hp+2kq-10lm+5nr)$$
$$+\tau(2gr+hi-jp+7ln-4mq)\}-*.$$

so that the whole is now a sum of three linear function of (ρ, σ, τ). $- *$.

73. Collecting the terms, we have

$$\partial_1\{0^2 13\}-\partial_0\{1^2 03\}=$$
$$\frac{1}{\Delta^3}\Big[\rho\Big\{(-8hn+9jm+2l^2-2qr)+\frac{1}{q}(4ghl-3gjr-2hij+j^2p+2jln)$$
$$+\frac{1}{q^2}(-2g^2h^2+4ghjn-2j^2n^2)\Big\}$$
$$+\sigma\Big\{\tfrac{1}{2}(-hp+kq-2lm+nr)+\frac{1}{q}(2hil-2hn^2-3ijr+jlp+2jmn)$$
$$+\frac{1}{q^2}(-2gh^2i+2ghln+2hijn-2jln^2)\Big\}$$
$$+\tau\Big\{\tfrac{1}{2}(gr+2hi+jp+mq+14ln)+\frac{1}{q}(-2ghn+2gjm-2ijl-2jn^2)$$
$$+\frac{1}{q^2}(2g^2hl-2ghij-2gjln+2ij^2n)\Big\}\Big]$$
$$-*.$$

The right-hand side depends on the points 0, 1, 3 and 2: viz. we have therein $\rho=023$, $\Delta=123$, etc., but the left-hand side depending on only the points 0, 1 and 3, the right-hand side cannot really contain 2, and it must thus remain unaltered, if for 2 we substitute any other point on the quartic, say 6: the right-hand side may therefore be understood as a function of 0, 1, 3 and 6,

viz. ρ, Δ, f, etc., will mean 063, 163, $6^3 3$, etc.: we have thus $\phi(0^1, 1, 3) =$ the above linear function with 2 thus replaced by 6; say

$$\phi(0^1, 1, 3) = \frac{1}{\varDelta^3}[\rho(\ ) + \sigma(\ ) + \tau(\ )],$$

a given function of the points 0, 1, 3 and the arbitrary point 6, on the quartic curve; we therefore write it $\phi(0^1, 1, 3, 6)$. There is no obvious value for $X(\overline{0, 1}, 3)$ which will produce any simplification, I therefore take this function to be $= 0$; and the final result is

$$Q_{0,12} = \{0^2 12\} + \Theta_{0,12} - (\{0^2 23\} + \{0^2 31\} + \{0^2 12\}),$$

where $\Theta_{0,12}$ is a function integral and linear as regards the coordinates (x, y, z) of the point 0, but transcendental as regards the parametric points 1, 2; and containing besides the arbitrary points 3, 6, of the quartic curve, its value being determined by the differential formulæ

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3, 6),\ \partial_2\Theta_{0,12} = -\phi(0^1, 2, 3, 6),$$

where $\phi(0^1, 1, 3, 6)$ is a given function as above. I do not see the meaning of the very complicated linear function of (ρ, σ, τ), nor how to reduce it to any form such as the simple one $\partial_3\{036\}$, which presents itself in the case of the cubic curve.

END OF CHAPTER III.

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